Non-Linear Wave-Oscillator Models for Transverse Vibrations of Offshore Structures

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Abstract

In this paper we will discuss the approximate solution of a two degree of freedom non-linear oscillator model subject to a simplified form for the description of wave and current loadings proposed in [SI81] by the averaging method. We will use this approximate solutions as basis for further analysis of nonlinear models in order to assess damping and loading terms that may give rise to ringing type response.

Introduction

A number of equations considered in the paper deal with the "ringing" problem which takes place in the offshore structure. Namely these are listed as following:

The non-linear dynamic problem of the interaction of the periodic motion of the vortexes in a current with an offshore structure is discussed and developed in the paper based on a well-known non-linear model presented in [SI81]. In order to solve the non-linear differential equations proposed to explain "ringing" problem of the offshore structure the averaging method [BM61] has been used to find the solution of the problem in question in the first approximation.

As a result a new method, based on *Mathematica* symbolic evaluations [Wol03] which convert the usual averaging procedure [BM61] into a computer routine, is developed for solving the nonlinear systems of differential equations.

The effects of vibrations on the offshore structure were also investigated and a mechanical balance modeling was performed to explain the experimental observations.

Non-linear vibration of the offshore structure under the disturbances produced by the flowing vortex is considered in detail when a number of resonances occur in the system.

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An averaging procedure is applied to the non-linear dynamic equations of the motion of the offshore structure to study the vibration of the structure under the load generated by the flow of the current.

In this paper *Mathematica* input is given in bold typefaces followed directly by the output whenever it exists, as the following example shows:

$\int_0^{\pi\tau} \sin\left(\frac{x}{3}\right) dx$

$6\sin^2\left(\frac{\pi\tau}{6}\right)$

The CPU timing has been done on a HP Compaq nx9420 computer with a dual core Intel Centrino 2.16GHz CPU, running *Mathematica* 6.0 and a SuSE Linux operating system.

The actual *Mathematica* 6.0 notebook, on which this paper is based, can be downloaded from `http://bluemaster.iu.hio.no/sims/pjg.nb`. The notebook-file contains the complete *Mathematica* code.

1. Dynamic Models of an Offshore Slender Structure

1.1. Equations of the Motion

The general second order ordinary differential equation for the horizontal response $y = y(t)$ of a one degree of freedom slender offshore structure when subjected to constant nonlinear *drag loading* (that is the *loading generated by the velocity of a current*) is according to experiments given per unit length of the structure; see for example Sarpkaya and Issacsson [SI81], is presented below.

The most noteworthy among the oscillator models is one proposed by Harlen and Currie (1970) [HC70] where a van der-Pol type-model soft non-linear oscillator for lift force is coupled to the body motion by a linear dependence on cylinder velocity.

This model is based partly on suggestion by Birkhopff and Zarantonello (1953) [BZ59] and by Bishop and Hassan (1963) [BH64] in connection with their experiments with oscillating cylinders in uniform flow.

The pair of equations which results from this concept are

$$\begin{aligned} \frac{d^2x_r}{dt^2} + 2\zeta_s\frac{dx_r}{dt} + x_r = a_0\Omega_0^2C_L \\ \frac{d^2C_L}{dt^2} - \alpha\Omega_0\frac{dC_L}{dt} + (\gamma/\Omega_0)\left(\frac{dC_L}{dt}\right)^3 + \Omega_0^2C_L = B\frac{dx_r}{dt} \end{aligned} \tag{1}$$

The dimensionless functions $x_r = (x/D_0)$ and $C_L = \frac{2F_L}{(\rho D_0 L V^2)}$ are the variables (dimensionless parameters) of the motion of the system.

The parameters α and γ are *the van der-Pol coefficients* and B is an interaction parameter.

Finally, ζ_s represents the material damping factor for the elastic system. There are no other damping imposed on the body.

Here the following desalinations are admitted:

$$
\begin{aligned}
\omega_n &= 2\pi f_n = \sqrt{k/m} \\
\Omega_0 &= f_0/f_n \\
\Omega_c &= f_c/f_n \\
\Omega_\nu &= f_\nu/f_n \\
\tau_n &= \omega_n t \\
S_0 &= \frac{f_0 D_0}{V} \\
x_r &= (x\,/D_0\,) \\
C_L &= \frac{2F_L}{(\rho D_0 L V^2)} \\
\hat{x}_r &= (A\,/D_0\,) \\
\hat{x}_{rm} &= (A\,/D_0\,)_{\max} \\
a_0 &= \frac{\rho D_0 L f_0^2}{(2mS_0^2\omega_n^2\Omega_0^2)} \\
\Delta_r &= \zeta_s/a_0 = (2\pi\zeta_s)\,(\pi S_0)^{\,2}/\rho_\tau \\
\rho_\tau &= \rho\,/\rho_s\, = 2\pi^3 a_0 S_0^2; \\
\omega_0 &= 2\pi f_0 \\
\omega_\nu &= 2\pi f_\vee \\
\omega_c &= 2\pi f_c
\end{aligned}
\tag{2}
$$

- m is the mass of the structure
- k is the linear stiffness of the structure
- c is a linear damper reaction coefficient
- D_0 is the diameter of the slender structure
- ρ is the density of the fluid (water)
- C_d is the drag coefficient for the flow
- $U(x,t)$ and $|U(x,t)|$ are the velocity and the absolute value of the velocity U of the constant flow (current) past the structure, x is the displacement.

This drag type loading is in general attributed to the shedding of vortices in the downstream flow direction of the current.

2. The General Method of Averaging

The main idea of the paper is to construct an approximate time depending solution of the non-linear system (1) at the time interval $[0, 1/\epsilon]$.

Usually the averaging method is to used to find out the first approximation solution of any non-linear dynamic system (1) converted in to a standard form. Then for the system in standard form the averaging procedure developed and proposed by N. N. Bogolubov - N. M. Krilov (see [BM61]) is applied.

To use this method, the system (1) is first reduced to the standard form.

$$\frac{d\overrightarrow{x}(t)}{dt} = \epsilon \overrightarrow{F}\left(\overrightarrow{x}(t), t\right) \tag{3}$$

where function $\overrightarrow{F}\left(\overrightarrow{x}(t), t\right)$ is T-periodic with respect to the second variable and ϵ is a small parameter.

Systems of the form (3) represent a classic topic of the theory of differential equations depending on ϵ - a small parameter and such systems have been developed by a number of various methods. For example, topological methods and vector field theory can be applied to study the system (3), (see, for instance, [AK74], [Kam96], [MS67]).

The following auxiliary system is considered based on an averaging procedure

$$\begin{aligned} \frac{d\overrightarrow{x}_0}{d\tau} &= \overrightarrow{F}\left(\overrightarrow{x}_0\right) \\ \tau &= \epsilon t \end{aligned} \tag{4}$$

where the function at the right side of (4), $\overrightarrow{F}\left(\overrightarrow{x}_0\right)$, is evaluated by the integral

$$\overrightarrow{F}\left(\overrightarrow{\zeta}\right) = (2\pi)^{-1}\int_0^{2\pi} \overrightarrow{F}\left(\overrightarrow{\zeta}, t\right) dt \tag{5}$$

Then a solution of the equation (3) as a series on the small parameter ϵ is written as following

$$\overrightarrow{x}(\tau) = \overrightarrow{x}_0(\tau) + \epsilon\overrightarrow{x}_1(\tau) + \epsilon^2\overrightarrow{x}_2(\tau) + \cdots \tag{6}$$

The isolated equilibrium states of system (4), which have non-zero topological index with respect to the vector field $\overrightarrow{F}\left(\overrightarrow{x}_0\right)$, give rise to T-periodic solutions of system (1). While the solutions of the Cauchy problem for system (1) are close to the corresponding solutions of the Cauchy problem for system (5) on the interval of length $1/\epsilon$.

The main method for reducing system (1) to standard form (2) consists in the following change of variables

$$\overrightarrow{z}(\tau) = f_s(\overrightarrow{x}(t), t, 0) \tag{7}$$

where $f_s(\overrightarrow{x}(t), t, 0)$ denotes the solution $\overrightarrow{x}(t)$ of (1) with the initial condition $\overrightarrow{x}(0) = \overrightarrow{X}_0$ and $\epsilon = 0$. Therefore, it is necessary to assume that the change of variables (7) is T-periodic with respect to t for every T-periodic function $\overrightarrow{x}(t)$ in order to use the classical averaging principle (see [Bai95], [Sch00]).

3. Linear Problem. Symbolic Solutions of the Linear Problem: Substitution Procedure

3.1. The Damped Motion of the Structure: The Full Interactive Linear Problem

Let us consider a linear problem of the motion of the structure which results from the non-linear problem (1) when one avoids all of the non-linear terms and terms corresponding to the damped forces in the system of differential equations (1).

Then a pair of linear differential equations resulting from the non-linear conceptual system (1) are presented below.

$$\begin{aligned} \frac{d^2x_r}{dt^2} + x_r = a_0\Omega_0^2 C_L \\ \frac{d^2C_L}{dt^2} + \Omega_0^2 C_L = B\frac{dx_r}{dt} \end{aligned} \tag{8}$$

This system (8) corresponds to the interactive and the free coupling motion of the rig and the vortex in the flow of the current.

There are a number of questions that follows from the linear problem (8):

- What is the motion of the offshore structure following from the linear model represented by equations (8)?
- Does any periodic solution exist in the linear system (8)?
- How to obtain from the system (8) any general solution which will be corresponded to the non-linear problem (1)?

In order to derive a general solution of the problem (8) a **DSolve** operator is used to find a symbolic solution of the linear problem (8).

A well-known procedure is proposed to solve the system (8) using step by step evaluation by the computer codes presented below.

sysInitial(X_, y_):= $\left\{X + \frac{d^2X}{dt^2} = a_0\Omega_0^2 y, y\Omega_0^2 + \frac{d^2y}{dt^2} = B\frac{dX}{dt}\right\}$;
$X = x_r(t); y = C_L(t);$

This code give us the following system of equations.

sysInitial$[X, y]$**//TraditionalForm**
$\left\{x_r(t) + x_r''(t) = a_0\Omega_0^2 C_L(t), C_L(t)\Omega_0^2 + C_L''(t) = Bx_r'(t)\right\}$

Then a general symbolic solution of the linear problem (8) is derived below

solSymbolic = DSolve [sysInitial$[X, y]$**,** $\{x_r[t], C_L[t]\}$ **,** t**] //Simplify//Flatten;**

The resulting *Mathematica* expression of **solSymbolic** is complex and contains the **RootSum** operator so the result is not shown here. No periodic solutions follows from the symbolic evaluations in clear shape of symbolic functions. If one consider an interactive problem with a small parameter of the interaction in the system when $B \ll 1$, (that physically means a small influence of the vortex on the motion of the rig), and if one transforms a symbolic solutions into periodic functions using a series expansions of the general solutions as below

generalSolution =
Series [$\{C_L(t), x_r(t)\}$ **/. solSymbolic,** $\{B, 0, 1\}$**] //Normal//ExpToTrig**
//ComplexExpand//Simplify//
TraditionalForm

$\Big\{ \frac{1}{2\Omega_0(\Omega_0^2-1)^2}\big(- c_3\cos(t\Omega_0)(Bta_0 - 2)\Omega_0^5 - \sin(t\Omega_0)(B(tc_2 - c_3)a_0 - 2c_2)\Omega_0^4$
$+ (2Bc_1\cos(t) - 2((Bc_1 + 2c_3)\cos(t\Omega_0) + Bc_4\sin(t)) + B(2c_2\cos(t)$
$+ (tc_3 - 2c_2)\cos(t\Omega_0) - 2c_3\sin(t))a_0)\Omega_0^3 + \sin(t\Omega_0)(-4c_2 + 2Bc_4$

$+ B(tc_2 + c_3)a_0)\Omega_0^2 + 2(-Bc_1\cos(t) + (Bc_1 + c_3)\cos(t\Omega_0) + Bc_4\sin(t))\Omega_0$
$+ 2(c_2 - Bc_4)\sin(t\Omega_0)\big),$
$\frac{1}{2(\Omega_0^2-1)^3}\big(Ba_0^2((tc_3\cos(t) + tc_3\cos(t\Omega_0) + (tc_2 + c_3)\sin(t))\Omega_0^3$
$+ (tc_2 - 3c_3)\sin(t\Omega_0)\Omega_0^2 - ((4c_2 + tc_3)\cos(t) + (tc_3 - 4c_2)\cos(t\Omega_0) + (tc_2 - 3c_3)\sin(t))\Omega_0 - (tc_2 + c_3)\sin(t\Omega_0))\Omega_0^3 - a_0(\Omega_0^2 - 1)(-((2c_3 + Btc_4)\cos(t) - 2c_3\cos(t\Omega_0) + (Btc_1 + 2c_2 - Bc_4)\sin(t))\Omega_0^3 + 2c_2\sin(t\Omega_0)\Omega_0^2 + ((2Bc_1 + 2c_3 + Btc_4)\cos(t) - 2(Bc_1 + c_3)\cos(t\Omega_0) + (Btc_1 + 2c_2 - 3Bc_4)\sin(t))\Omega_0 - 2(c_2 - Bc_4)\sin(t\Omega_0))\Omega_0 + 2(c_4\cos(t) + c_1\sin(t))(\Omega_0^2 - 1)^3\big)\Big\}$

then unstable motion of the structure ($\{C_L(t), x_r(t)\}\to\infty$, when $t\to\infty$) are resulting from the symbolic expansions of the general solutions of the system (9) obtained above. Hence the general solution of the linear system (8) can not be used as a general solution of the system for the non-linear model (1). We conclude that another general linear system derived from (1) has to be derived for further evaluations.

4. Computer Procedure for Converting ODE to Standard Form

4.1. No damped Motion of the Structure: Semi interactive problem

In order to study the physical sense of the problem in question let us consider a linear semi interactive problem that means the act of the motion of the vortex on rig and no response from the rig to the motion of the vortex. A mathematical formulation of such a mechanical problem corresponds to the problem described by the following differential equations.

$$\begin{aligned}&\frac{d^2x_r}{dt^2} + x_r = a_0\Omega_0^2C_L\\&\frac{d^2C_L}{dt^2} + \Omega_0^2C_L = 0\end{aligned}\tag{9}$$

Here a pair of linear differential equations is derived from the system (8) when $B\to 0$. The mechanical explanation of such a suggestion is explained above.

Now let us consider a general solution of the system (9) .

solSymbolic = DSolve [sysInitial[X, y]/.$B\to$ 0, {x_r[t], C_L[t]} , t]
//Simplify//Flatten;

A full form of the general symbolic solutions of the linear problem (9) is presented below.

(generalSolution = solSymbolic//Factor)//TraditionalForm

$$\Big\{C_L(t)\to\frac{c_2\sin(t\Omega_0) + c_3\cos(t\Omega_0)\Omega_0}{\Omega_0},$$
$$x_r(t)\to\frac{1}{(\Omega_0-1)(\Omega_0+1)}\big(c_4\cos(t)\Omega_0^2 + c_1\sin(t)\Omega_0^2 + c_3\cos(t)a_0\Omega_0^2$$
$$-c_3\cos(t\Omega_0)a_0\Omega_0^2 + c_2\sin(t)a_0\Omega_0^2 - c_2\sin(t\Omega_0)a_0\Omega_0 - c_4\cos(t) - c_1\sin(t)\big)\Big\}$$

A periodic general solution of the semi interactive system will be chosen as a generated solution for the system of non-linear equations (1).

4.2. Standard Form of the System of Equations

According to the general averaging procedure let us convert a general symbolic solution of the semi interactive linear equations (9) to the general standard form, equations (3).

Here the slow time variables $c_i(\tau), (\tau = \epsilon t)$ are introduced by the formulas

$$\begin{aligned} C_L(t,\tau) &= \frac{\sin(t\Omega_0)c_2(\tau)}{\Omega_0} + \cos(t\Omega_0)c_3(\tau) \\ x_r(t) &= \frac{1}{\Omega_0^2 - 1}(a_0\Omega_0((c_3\cos(t) - c_3\cos(t\Omega_0) + c_2\sin(t))\Omega_0 \\ &\quad - c_2\sin(t\Omega_0)) + (c_4\cos(t) + c_1\sin(t))(\Omega_0^2 - 1)) \end{aligned} \tag{10}$$

Codes for the variables $C_L(t,\tau)$ and $x_r(t,\tau)$ in the slow time variable, given as in (7), are written as below

(substitutionVortex = Part[generalSolution, 1]/.{$C_L(t) \to C_L(t,\tau), c_2 \to c_2(\tau)$, $c_3 \to c_3(\tau)$ })//TraditionalForm

$$C_L(t,\tau) \to \frac{\sin(t\Omega_0)\,c_2(\tau) + \cos(t\Omega_0)\,\Omega_0 c_3(\tau)}{\Omega_0}$$

(substitutionStructure =
Part[generalSolution, 2]/. {$x_r(t) \to x_r(t,\tau), c_1 \to c_1(\tau), c_2 \to c_2(\tau), c_3 \to c_3(\tau)$, $c_4 \to c_4(\tau)$ }) //TraditionalForm

$$x_r(t,\tau) \to \frac{1}{(\Omega_0-1)(\Omega_0+1)}(\sin(t)c_1(\tau)\Omega_0^2 + \sin(t)a_0c_2(\tau)\Omega_0^2 + \cos(t)a_0c_3(\tau)\Omega_0^2 - \cos(t\Omega_0)a_0c_3(\tau)\Omega_0^2 + \cos(t)c_4(\tau)\Omega_0^2 - \sin(t\Omega_0)a_0c_2(\tau)\Omega_0 - \sin(t)c_1(\tau) - \cos(t)c_4(\tau))$$

Symbolic derivatives for $\dfrac{dC_L(t,\tau)}{dt}$ and $\dfrac{dx_r(t,\tau)}{dt}$ are accounted in symbolic form by the operators

(substitutionVelocityVortex =
D[Part[substitutionVortex, 2], t]) //TraditionalForm

$$\frac{\cos(t\Omega_0)\,\Omega_0 c_2(\tau) - \sin(t\Omega_0)\,\Omega_0^2 c_3(\tau)}{\Omega_0}$$

(substitutionVelocityStructure = D[Part[substitutionStructure, 2], t])
//TraditionalForm

$$\begin{aligned} &\frac{1}{(\Omega_0-1)(\Omega_0+1)}(\sin(t\Omega_0)a_0c_3(\tau)\Omega_0^3 + \cos(t)c_1(\tau)\Omega_0^2 \\ &+ \cos(t)a_0c_2(\tau)\Omega_0^2 - \cos(t\Omega_0)a_0c_2(\tau)\Omega_0^2 - \sin(t)a_0c_3(\tau)\Omega_0^2 \\ &- \sin(t)c_4(\tau)\Omega_0^2 - \cos(t)c_1(\tau) + \sin(t)c_4(\tau)) \end{aligned}$$

Then other substitutions for $C_L(t)$ and $\dfrac{dC_L(t,\tau)}{dt}$ are presented as following

$$\begin{aligned} C_L(t,\tau) &= c_3(\tau)\cos(t\Omega_0) + \frac{c_2(\tau)\sin(t\Omega_0)}{\Omega_0} \\ \frac{dC_L(t,\tau)}{dt} &= c_2(\tau)\cos(t\Omega_0) - c_3(\tau)\Omega_0\sin(t\Omega_0) \end{aligned} \tag{11}$$

In order to find a full set of equations for the new set of slowly variables $\{c_1 \to c_1(\tau), c_2 \to c_2(\tau), c_3 \to c_3(\tau), c_4 \to c_4(\tau)\}$, one has to construct a new system of equations regarding the new variables $\{c_1(\tau), c_2(\tau), c_3(\tau), c_4(\tau)\}$ and their derivatives.

$$C_L'(t,\tau)\big|_{\tau\to t} = \frac{dC_L(t,\tau)}{dt} \tag{12}$$

Then a first standard form of equations are developed from (11) and (12) and by codes the below

(eq1 = ((*D*[Part[substitutionVortex, 2]/.*t* → *τ*, *τ*]/.{cos(*τ*Ω₀) → cos(*t*Ω₀), sin(*τ*Ω₀) → sin(*t*Ω₀)}) == (substitutionVelocityVortex))//Simplify) //TraditionalForm

$$\frac{\sin(t\Omega_0)\,c_2'(\tau)}{\Omega_0} + \cos(t\Omega_0)\,c_3'(\tau) = 0$$

Substitutions for $x_r(t,\tau)$ and $\dfrac{dx_r(t,\tau)}{dt}$ are proposed as following

$$\begin{aligned} x_r(t,\tau) &= \frac{1}{\Omega_0^2-1}(a_0\Omega_0(\Omega_0(\sin(t)c_2(\tau)+\cos(t)c_3(\tau)-\cos(t\Omega_0)c_3(\tau)) \\ &\quad -\sin(t\Omega_0)c_2(\tau)) + (\Omega_0^2-1)(\sin(t)c_1(\tau)+\cos(t)c_4(\tau))) \\ \frac{dx_r(t,\tau)}{dt} &= \frac{1}{\Omega_0^2-1}(a_0((\cos(t)-\cos(t\Omega_0))c_2(\tau)+(\sin(t\Omega_0)\Omega_0 \\ &\quad -\sin(t))c_3(\tau))\Omega_0^2 + \cos(t)(\Omega_0^2-1)c_1(\tau) - \sin(t)(\Omega_0^2-1)c_4(\tau)) \end{aligned} \tag{13}$$

The second step of the evaluation consists of finding the equality from (13)

$$x_r'(t,\tau) = \frac{dx_r(t,\tau)}{dt} \tag{14}$$

So the second equation for the standard form of the system is derived by the code as following

(eq2 = ((*D*[Part[substitutionStructure, 2]/.*t* → *τ*, *τ*]/.{cos(*τ*Ω₀) → cos(*t*Ω₀), sin(*τ*Ω₀) → sin(*t*Ω₀), cos(*τ*) → cos(*t*), sin(*τ*) → sin(*t*)}) == (substitutionVelocityStructure))//Simplify)//TraditionalForm

$$\tfrac{1}{\Omega_0^2-1}(\sin(t)(\Omega_0^2-1)c_1'(\tau) + a_0\Omega_0((\sin(t)\Omega_0 \\ -\sin(t\Omega_0))c_2'(\tau) + (\cos(t)-\cos(t\Omega_0))\Omega_0 c_3'(\tau)) + \cos(t)(\Omega_0^2-1)c_4'(\tau)) = 0$$

The initial system of differential equations (1) is reduced to the standard form of equations

$$\begin{aligned} \frac{d^2X}{dt^2} + X &= a_0\Omega_0^2 y + \epsilon\left(-2\zeta_s\frac{dX}{dt}\right) \\ \frac{d^2y}{dt^2} + \Omega_0^2 y &= \epsilon\left(B\frac{dX}{dt} + \alpha\Omega_0\frac{dy}{dt} - (\gamma/\Omega_0)\left(\frac{dy}{dt}\right)^3\right) \end{aligned} \tag{15}$$

as well the same system (15) in codes is given to the standard form of equations.

**(eq3 =
((*D*[substitutionVelocityVortex/.*t* → *τ*, *τ*]/.{cos(*τ*Ω₀) → cos(*t*Ω₀),**

$\sin(\tau\Omega_0) \to \sin(t\Omega_0)\}) + \Omega_0^2$ Part[substitutionVortex, 2] ==
$\varepsilon(B$(substitutionVelocityStructure) + $\alpha\Omega_0$(substitutionVelocityVortex)
$-(\gamma/\Omega_0)$(substitutionVelocityVortex)^3))//Simplify)//TraditionalForm

$$\cos(t\Omega_0)c_2'(\tau) = \varepsilon\Bigg(\frac{\gamma(\sin(t\Omega_0)\Omega_0 c_3(\tau) - \cos(t\Omega_0)c_2(\tau))^3}{\Omega_0} - \alpha\Omega_0(\sin(t\Omega_0)\Omega_0 c_3(\tau) - \cos(t\Omega_0)c_2(\tau)) + \frac{1}{\Omega_0^2-1}B\big(a_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) + (\sin(t\Omega_0)\Omega_0 - \sin(t))c_3(\tau)\big)\Omega_0^2 + \cos(t)(\Omega_0^2 - 1)c_1(\tau) - \sin(t)(\Omega_0^2 - 1)c_4(\tau))\Bigg) + \sin(t\Omega_0)\Omega_0 c_3'(\tau)$$

(eq4 =
((D[substitutionVelocityStructure/.$t \to \tau, \tau$]/.{$\cos(\tau\Omega_0) \to \cos(t\Omega_0)$,
$\sin(\tau\Omega_0) \to \sin(t\Omega_0), \cos(\tau) \to \cos(t), \sin(\tau) \to \sin(t)\}$)
+Part[substitutionStructure, 2] == $a_0\Omega_0^2$Part[substitutionVortex, 2]
+$\varepsilon(-2\zeta_s$substitutionVelocityStructure))//Simplify)//TraditionalForm

$$\frac{1}{\Omega_0^2-1}(\sin(t\Omega_0)a_0c_3'(\tau)\Omega_0^3 + \cos(t)a_0c_2'(\tau)\Omega_0^2 - \cos(t\Omega_0)a_0c_2'(\tau)\Omega_0^2 - \sin(t)a_0c_3'(\tau)\Omega_0^2 - \sin(t)c_4'(\tau)\Omega_0^2 + 2\varepsilon\zeta_s(a_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) + (\sin(t\Omega_0)\Omega_0 - \sin(t))c_3(\tau))\Omega_0^2 + \cos(t)(\Omega_0^2 - 1)c_1(\tau) - \sin(t)(\Omega_0^2 - 1)c_4(\tau)) + \cos(t)(\Omega_0^2 - 1)c_1'(\tau) + \sin(t)c_4'(\tau)) = 0$$

Finally, the following system of differential equations regarding the slowly variables $c_i(\tau)$ written in the "slow" time τ is presented in the standard form of the differential equations (3).

(sol1 = Solve[{eq1, eq2, eq3, eq4}, $\{c_1'(\tau), c_2'(\tau), c_3'(\tau), c_4'(\tau)\}$]
//Flatten//Simplify) //TraditionalForm

$$\Big\{c_1'(\tau) \to -\frac{1}{8(\Omega_0^2-1)^2}\varepsilon(16\cos(t)\zeta_s(\Omega_0^2-1)(a_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) + (\sin(t\Omega_0)\Omega_0 - \sin(t))c_3(\tau))\Omega_0^2 + \cos(t)(\Omega_0^2 - 1)c_1(\tau) - \sin(t)(\Omega_0^2 - 1)c_4(\tau)) - 2(\cos(t) - \cos(t\Omega_0))a_0\Omega_0(4\sin(t\Omega_0)c_3(\tau)(\gamma\sin^2(t\Omega_0)c_3(\tau)^2 - \alpha)\Omega_0^5 + (4B\sin(t\Omega_0)a_0c_3(\tau) + 4\cos(t\Omega_0)c_2(\tau)(\alpha - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2))\Omega_0^4 + (-3\gamma\sin(t\Omega_0)c_3(\tau)^3 + \gamma\sin(3t\Omega_0)c_3(\tau)^3 + 3\gamma\sin(t\Omega_0)c_2(\tau)^2c_3(\tau) + 3\gamma\sin(3t\Omega_0)c_2(\tau)^2c_3(\tau) + 4\alpha\sin(t\Omega_0)c_3(\tau) + 4B\cos(t)c_1(\tau) + 4Ba_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) - \sin(t)c_3(\tau)) - 4B\sin(t)c_4(\tau))\Omega_0^3 - 4\cos(t\Omega_0)c_2(\tau)(\gamma\cos^2(t\Omega_0)c_2(\tau)^2 - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2 + \alpha)\Omega_0^2 - 4(3\gamma\cos^2(t\Omega_0)\sin(t\Omega_0)c_3(\tau)c_2(\tau)^2 + B\cos(t)c_1(\tau) - B\sin(t)c_4(\tau))\Omega_0 + 4\gamma\cos^3(t\Omega_0)c_2(\tau)^3)),$$

$$c_4'(\tau) \to \frac{1}{8(\Omega_0^2-1)^2}\varepsilon(16\sin(t)\zeta_s(\Omega_0^2-1)(a_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) + (\sin(t\Omega_0)\Omega_0 - \sin(t))c_3(\tau))\Omega_0^2 + \cos(t)(\Omega_0^2 - 1)c_1(\tau) - \sin(t)(\Omega_0^2 - 1)c_4(\tau)) + 2a_0(\sin(t\Omega_0) - \sin(t)\Omega_0)(4\sin(t\Omega_0)c_3(\tau)(\gamma\sin^2(t\Omega_0)c_3(\tau)^2 - \alpha)\Omega_0^5 + (4B\sin(t\Omega_0)a_0c_3(\tau) + 4\cos(t\Omega_0)c_2(\tau)(\alpha - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2))\Omega_0^4 + (-3\gamma\sin(t\Omega_0)c_3(\tau)^3 + \gamma\sin(3t\Omega_0)c_3(\tau)^3 + 3\gamma\sin(t\Omega_0)c_2(\tau)^2c_3(\tau) + 3\gamma\sin(3t\Omega_0)c_2(\tau)^2c_3(\tau) + 4\alpha\sin(t\Omega_0)c_3(\tau) + 4B\cos(t)c_1(\tau) + 4Ba_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) - \sin(t)c_3(\tau)) - 4B\sin(t)c_4(\tau))\Omega_0^3 - 4\cos(t\Omega_0)c_2(\tau)(\gamma\cos^2(t\Omega_0)c_2(\tau)^2 - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2 + \alpha)\Omega_0^2 - 4(3\gamma\cos^2(t\Omega_0)\sin(t\Omega_0)c_3(\tau)c_2(\tau)^2 + B\cos(t)c_1(\tau) - B\sin(t)c_4(\tau))\Omega_0 + 4\gamma\cos^3(t\Omega_0)c_2(\tau)^3)),$$

$$c_2'(\tau) \to \frac{1}{\Omega_0(\Omega_0^2-1)}\varepsilon\cos(t\Omega_0)((\gamma\sin^3(t\Omega_0)c_3(\tau)^3 - \alpha\sin(t\Omega_0)c_3(\tau))\Omega_0^5 + (B\sin(t\Omega_0)a_0c_3(\tau) + \cos(t\Omega_0)c_2(\tau)(\alpha - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2))\Omega_0^4 + (-\gamma\sin^3(t\Omega_0)c_3(\tau)^3 + 3\gamma\cos^2(t\Omega_0)\sin(t\Omega_0)c_2(\tau)^2c_3(\tau) + \alpha\sin(t\Omega_0)c_3(\tau) + B\cos(t)c_1(\tau) + Ba_0((\cos(t) - \cos(t\Omega_0))c_2(\tau) - \sin(t)c_3(\tau)) - B\sin(t)c_4(\tau))\Omega_0^3 - \cos(t\Omega_0)c_2(\tau)(\gamma\cos^2(t\Omega_0)c_2(\tau)^2 - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2 + \alpha)\Omega_0^2$$

$+ (-3\gamma \cos^2(t\Omega_0)\sin(t\Omega_0)c_3(\tau)c_2(\tau)^2 - B\cos(t)c_1(\tau) + B\sin(t)c_4(\tau))\Omega_0$
$+ \gamma\cos^3(t\Omega_0)c_2(\tau)^3),$
$c_3'(\tau) \to \frac{1}{\Omega_0^2(\Omega_0^2-1)}\varepsilon\sin(t\Omega_0)(\sin(t\Omega_0)c_3(\tau)(\alpha - \gamma\sin^2(t\Omega_0)c_3(\tau)^2)\Omega_0^5$
$-(B\sin(t\Omega_0)a_0c_3(\tau) + \cos(t\Omega_0)c_2(\tau)(\alpha - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2))\Omega_0^4$
$+ (\gamma\sin^3(t\Omega_0)c_3(\tau)^3 - 3\gamma\cos^2(t\Omega_0)\sin(t\Omega_0)c_2(\tau)^2c_3(\tau) - \alpha\sin(t\Omega_0)c_3(\tau)$
$-B\cos(t)c_1(\tau) + Ba_0((\cos(t\Omega_0) - \cos(t))c_2(\tau) + \sin(t)c_3(\tau)) + B\sin(t)c_4(\tau))\Omega_0^3 +$
$\cos(t\Omega_0)c_2(\tau)(\gamma\cos^2(t\Omega_0)c_2(\tau)^2 - 3\gamma\sin^2(t\Omega_0)c_3(\tau)^2 + \alpha)\Omega_0^2$
$+ (3\gamma\cos^2(t\Omega_0)\sin(t\Omega_0)c_3(\tau)c_2(\tau)^2 + B\cos(t)c_1(\tau) - B\sin(t)c_4(\tau))\Omega_0$
$-\gamma\cos^3(t\Omega_0)c_2(\tau)^3)\Big\}$

4.3. Averaging Procedure of Evaluation in *Mathematica*

In order to obtain a system of differential equations of the first approximation in the form (4) the following main code of symbolic integration in the procedural complex code is applied. The resulting output takes several pages and is omitted here. The **Timing** operator also shows that the computation requires a lot of CPU time.

(system = Table[Apply[Equal, Rule[Part[sol1, i, 1],
$(2\pi)^{-1}$Integrate[Part[sol1, i, 2], {t, 0, 2π}]]], {i, 1, 4}]//Simplify); //Timing

{241.767, Null}

As a result an averaging system of the first approximation in the form of (4) is obtained in the set of the new variables $\{c_i[\tau]\}$ and in a standard form of evaluation.

4.4. Steady State Solutions

Firstly let us consider a set of the steady state positions of the system derived from the equations of the first approximation given by the code above. A resonance condition in the system is taken into consideration.

A set of stationery points of the non-linear averaging system implies by operator

$$\epsilon(2\pi)^{-1}\int_0^{2\pi}\vec{F}\left(\vec{x}_0, t\right)\,dt = 0 \qquad (16)$$

A computer code to find a set of the parameters of the steady state positions (points) of the system (16) is presented below.

A symbolic solution for the subharmonic resonance is derived below. Again the output takes several pages and is not shown here. The computation also requires some CPU time.

(solutionStationery = Solve[system/.{$c_1'(\tau) \to 0, c_2'(\tau) \to 0, c_3'(\tau) \to 0,$
$c_4'(\tau) \to 0$}/.{$\Omega_0 \to 1/2$}, {$c_1[\tau], c_2[\tau], c_3[\tau], c_4[\tau]$}]
//Chop//Flatten//Simplify); //Timing

{33.1781, Null}

A numerical solution is given below.

(solRingingMianResonance =
Solve[system/.{$c_1'(\tau) \to 0, c_2'(\tau) \to 0, c_3'(\tau) \to 0, c_4'(\tau) \to 0$}/.{$\zeta_s \to .001$,

$\boldsymbol{a_0 \to .1, \Omega_0 \to 1.001, \alpha \to 1.1, \gamma \to .1, B \to .2, \varepsilon \to 1\},}$
$\boldsymbol{\{c_1[\tau], c_2[\tau], c_3[\tau], c_4[\tau]\}]}$**//Chop//Flatten//Simplify)//TraditionalForm**

$\{c_1[\tau] \to -11.511 - 1.21678i, c_4[\tau] \to 0.065096 - 215.135i, c_2[\tau] \to 0.414104+0.024299i, c_3[\tau] \to -0.00129986+7.72693i, c_1[\tau] \to -11.511+1.21678i, c_4[\tau] \to 0.065096 + 215.135i, c_2[\tau] \to 0.414104 - 0.024299i, c_3[\tau] \to - 0.00129986 - 7.72693i, c_1[\tau] \to -3.93203, c_4[\tau] \to 0.533678, c_2[\tau] \to 3.39579, c_3[\tau] \to -0.0106576, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 0, c_1[\tau] \to -1.24727i, c_4[\tau] \to -230.039i, c_2[\tau] \to 0.0249083i, c_3[\tau] \to 7.92045i, c_1[\tau] \to 1.24727i, c_4[\tau] \to 230.039i, c_2[\tau] \to -0.0249083i, c_3[\tau] \to -7.92045i, c_1[\tau] \to 3.93203, c_4[\tau] \to -0.533678, c_2[\tau] \to -3.39579, c_3[\tau] \to 0.0106576, c_1[\tau] \to 11.511 - 1.21678i, c_4[\tau] \to -0.065096 - 215.135i, c_2[\tau] \to -0.414104 + 0.024299i, c_3[\tau] \to 0.00129986 + 7.72693i, c_1[\tau] \to 11.511 + 1.21678i, c_4[\tau] \to -0.065096 + 215.135i, c_2[\tau] \to -0.414104 - 0.024299i, c_3[\tau] \to 0.00129986 - 7.72693i\}$

Then the numerical solutions according to a steady state motion for resonance conditions $\Omega_0 \to 1/2$ occurring in a dynamic system are presented below.

(solRingingSubHarmonic =
Solve[system/.$\boldsymbol{\{c_1'(\tau) \to 0, c_2'(\tau) \to 0, c_3'(\tau) \to 0, c_4'(\tau) \to 0\}/.\{\zeta_s \to .001,}$
$\boldsymbol{a_0 \to .1, \Omega_0 \to 1/2, \alpha \to 1.1, \gamma \to .1, B \to .2, \varepsilon \to 1\},}$
$\boldsymbol{\{c_1[\tau], c_2[\tau], c_3[\tau], c_4[\tau]\}]}$**//Chop//Flatten//Simplify)//TraditionalForm**

$\{c_1[\tau] \to -0.057337, c_4[\tau] \to -1.14342, c_2[\tau] \to -1.72011, c_3[\tau] \to 0, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to -4.05223, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 0, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to -0.814897i, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to -0.814897i, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 0.814897i, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 0.814897i, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 4.05223, c_1[\tau] \to 0.057337, c_4[\tau] \to 1.14342, c_2[\tau] \to 1.72011, c_3[\tau] \to 0\}$

A set of other points in the superharmonic resonance condition is obtained by the operator below.

(solRingingSuperHarmonic =
Solve[system/.$\boldsymbol{\{c_1'(\tau) \to 0, c_2'(\tau) \to 0, c_3'(\tau) \to 0, c_4'(\tau) \to 0\}/.\{\zeta_s \to .04,}$
$\boldsymbol{a_0 \to .1, \Omega_0 \to 2, \alpha \to 1.1, \gamma \to .1, B \to .2, \varepsilon \to 1\},}$
$\boldsymbol{\{c_1[\tau], c_2[\tau], c_3[\tau], c_4[\tau]\}]}$**//Chop//Flatten//Simplify)//TraditionalForm**

$\{c_1[\tau] \to -1.01505, c_4[\tau] \to 0, c_2[\tau] \to 7.61285, c_3[\tau] \to 0, c_1[\tau] \to 0, c_4[\tau] \to -0.507524, c_2[\tau] \to 0, c_3[\tau] \to 3.80643, c_1[\tau] \to 0, c_4[\tau] \to -0.507524, c_2[\tau] \to 0, c_3[\tau] \to 3.80643, c_1[\tau] \to 0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 0, c_1[\tau] \to 0, c_4[\tau] \to 0.507524, c_2[\tau] \to 0, c_3[\tau] \to -3.80643, c_1[\tau] \to 0, c_4[\tau] \to 0.507524, c_2[\tau] \to 0, c_3[\tau] \to -3.80643, c_1[\tau] \to 1.01505, c_4[\tau] \to 0, c_2[\tau] \to -7.61285, c_3[\tau] \to 0\}$

(solRingingSuperHarmonic =
Solve[system/.$\boldsymbol{\{c_1'(\tau) \to 0, c_2'(\tau) \to 0, c_3'(\tau) \to 0, c_4'(\tau) \to 0\}/.}$
$\boldsymbol{\{\zeta_s \to .0001, a_0 \to .1, \Omega_0 \to 1/3 + .001, \alpha \to 1.1, \gamma \to .1, B \to .2, \varepsilon \to 1\},}$
$\boldsymbol{\{c_1[\tau], c_2[\tau], c_3[\tau], c_4[\tau]\}]}$**//Chop//Flatten//**
Simplify)//TraditionalForm

$\{c_1[\tau] \to -0.753868, c_4[\tau] \to -0.058738, c_2[\tau] \to -1.28772, c_3[\tau] \to -0.0978205, c_1[\tau] \to -0.729116, c_4[\tau] \to 0.0987087, c_2[\tau] \to 0.679076, c_3[\tau] \to 3.27399, c_1[\tau] \to -0.72342, c_4[\tau] \to -0.041904, c_2[\tau] \to 0.638087, c_3[\tau] \to -3.32979, c_1[\tau] \to -0.0144799, c_4[\tau] \to -0.739561, c_2[\tau] \to -1.15061, c_3[\tau] \to -1.97256, c_1[\tau] \to$

$0, c_4[\tau] \to 0, c_2[\tau] \to 0, c_3[\tau] \to 0, c_1[\tau] \to 0.0144799, c_4[\tau] \to 0.739561, c_2[\tau] \to 1.15061, c_3[\tau] \to 1.97256, c_1[\tau] \to 0.72342, c_4[\tau] \to 0.041904, c_2[\tau] \to -0.638087, c_3[\tau] \to 3.32979, c_1[\tau] \to 0.729116, c_4[\tau] \to -0.0987087, c_2[\tau] \to -0.679076, c_3[\tau] \to -3.27399, c_1[\tau] \to 0.753868, c_4[\tau] \to 0.058738, c_2[\tau] \to 1.28772, c_3[\tau] \to 0.0978205\}$

5. Numerical Simulation for the Main Resonance Condition.

Let us consider a numerical solution of the system of the first approximations when a *main resonance* condition is taking place in the system. The damping parameter of the rig ζ_s is the single parameter being varied in the system.

The structural damping ζ_s is associated with a constant parameter of the offshore structure. As is well known in structural analysis, the damping term changes the natural frequency of motion of the structure and the general motion of the system.

A code for numerical solution is presented below with an initial conditions taken very close to the steady state position.

It should be noted that the resulting solution as function of slowly time should be analyzed further to check whether other solution schemes, possibly with other values of the structural damping parameter ζ_s, would reveal different kind of solutions to the problem.

solNumerical =
NDSolve[Join[system/.{$\zeta_s \to .0191, a_0 \to .1, \Omega_0 \to 1.001, \alpha \to 1.1, \gamma \to .1, B \to .2, \varepsilon \to 1\}, \{c_1(0) == -.06410, c_2(0) == -1.9, c_3(0) == 0.0, c_4(0) == -.120\}], \{c_1(\tau), c_2(\tau, c_3(\tau), c_4(\tau)\}, \{\tau, 0, 2900\}$]//Flatten;

The numeric solution of the ODE in the first approximation with the structural damping parameter $\zeta_s = 0.0191$ given in [SI81] reveals a solution that is quite similar to the solution found by numerical integration of the exact solution in [JG06].

A stable motion of the structure given in the phase space of the slowly varying parameters of the amplitudes is presented in figure 1 to figure 5 below. The wave's motion of the vortex is shown in figure 3 and in figure 4.
The graphics in figure 5 shows the trajectory of the curve $(c_1(\tau), c_3(\tau), c_4(\tau))$ for $\tau \in [10, 2800]$. The trajectory approaches a stationary point.
As a result of the simulation of the structure with main resonance condition in the system a set of the stable amplitudes of vibrations are obtained.

A numeric simulation of the system when $\zeta_s < 0.0191$ based on the code below is given in figure 6 to figure 10.

solNumerical =
NDSolve[Join[system/.{$\zeta_s \to .001, a_0 \to .1, \Omega_0 \to 1.001, \alpha \to 1.1, \gamma \to .1, B \to .2, \varepsilon \to 1\}, \{c_1(0) == -.0641, c_2(0) == -1.9, c_3(0) == .0, c_4(0) == -.120\}], \{c_1(\tau), c_2(\tau), c_3(\tau), c_4(\tau)\}, \{\tau, 0, 2900\}$]//Flatten;

We will now find other kind of motion of the structure given by the numerical form of the solution for the equations of the first approximation. Examples of the non-stationary motion of the specific point in the phase space are presented in figure 6 to figure 10.

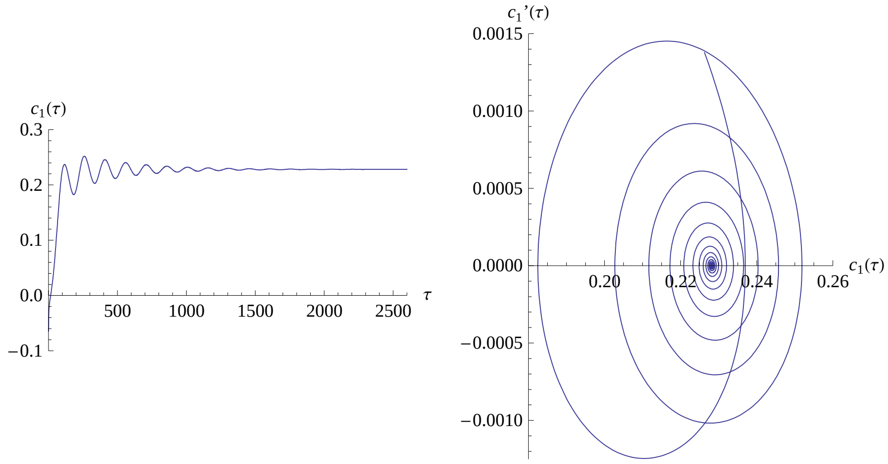

Figure 1: Stable motion of $c_1(\tau)$ to a stationary point. Here $\zeta_s = 0.0191$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_1(\tau)$ as a function of time, on the right, the phasespace of (c_1, c_1').

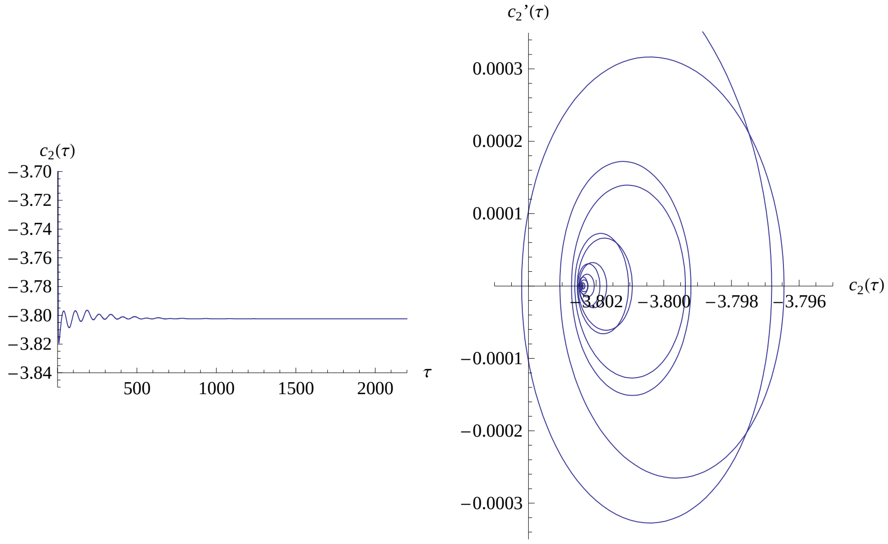

Figure 2: Stable motion of $c_2(\tau)$ to a stationary point. Here $\zeta_s = 0.0191$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_2(\tau)$ as a function of time, on the right, the phasespace of (c_2, c_2').

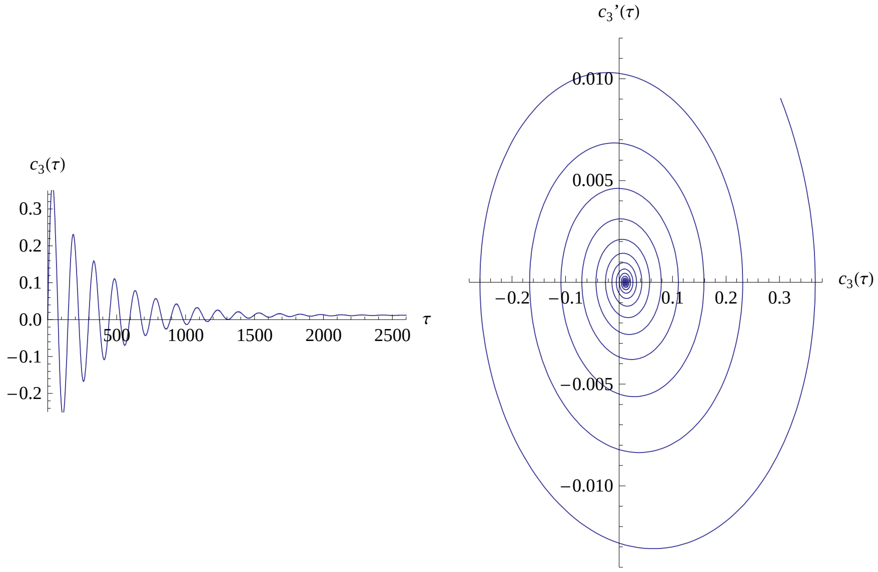

Figure 3: The wave's motion of the vortex. Stable motion of $c_3(\tau)$ to a stationary point. Here $\zeta_s = 0.0191$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_3(\tau)$ as a function of time, on the right, the phasespace of (c_3, c_3').

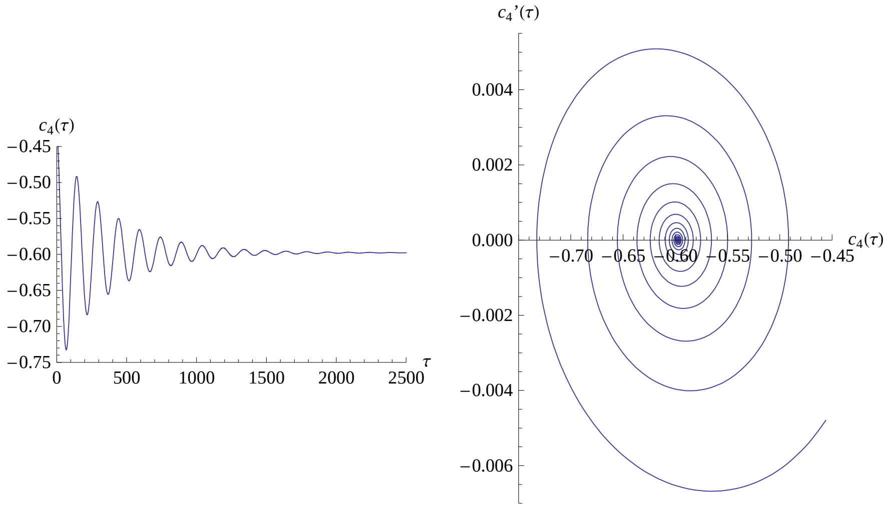

Figure 4: The wave's motion of the vortex. Stable motion of $c_4(\tau)$ to a stationary point. Here $\zeta_s = 0.0191$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_4(\tau)$ as a function of time, on the right, the phasespace of (c_4, c_4').

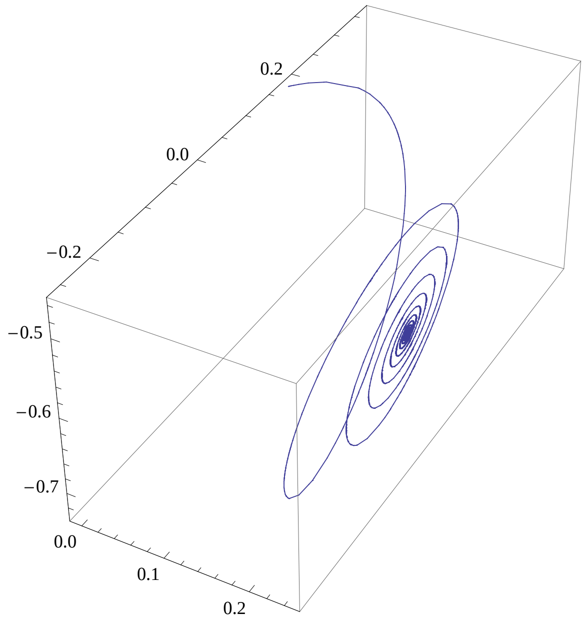

Figure 5: The trajectory of the projection of a curve in 4-space to 3-space, $\pi_{1,3,4}c(\tau) = (c_1(\tau), c_3(\tau), c_4(\tau))$ for $\tau \in [10, 2800]$ for $\zeta_s = 0.0191$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. Since each componentfunction tends to a stationary solution, the trajectory of $\tau \rightarrow c(\tau)$ also approaches a stationary point.

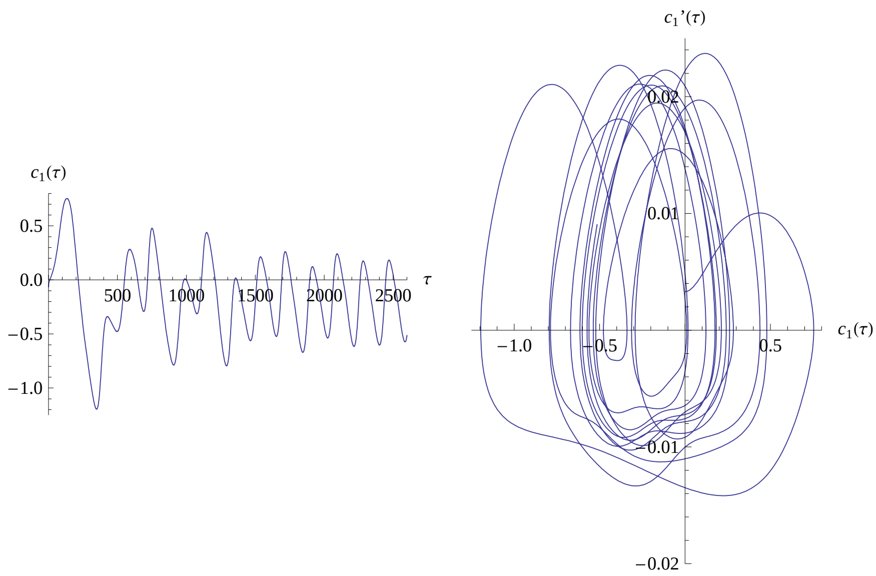

Figure 6: Quasiperiodic motion of $c_1(\tau)$. Here $\zeta_s = 0.001$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_1(\tau)$ as a function of time, on the right, the phasespace of (c_1, c_1').

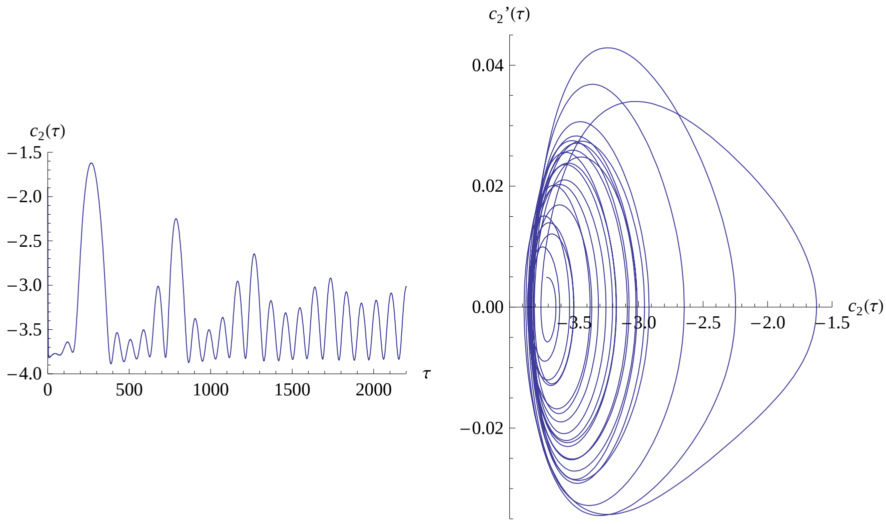

Figure 7: Quasiperiodic motion of $c_2(\tau)$. Here $\zeta_s = 0.001$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_2(\tau)$ as a function of time, on the right, the phasespace of (c_2, c_2').

The wave's motion of the vortex is shown in figure 8 and in figure 9.
Figure 10 shows the trajectory of the curve $(c_1(\tau), c_2(\tau), c_4(\tau))$ for $\tau \in [100, 2800]$. It is not clear from this simulation what kind of behaviour the trajectory has.

The non-stable amplitudes of the vibrations of the rig together with vortex one observes in the phase diagrams presented above. Reducing the structural damping coefficient ζ_s associated with the structural damping of the rig, leads the dynamic system to the non stationary vibrations with amplitudes depending on the slowly time τ.

6. Subharmonic Resonance 1/2

Now let us consider a numerical solutions of the system of the first approximations when a *subharmonic resonance* condition is taking place in the system. The damping parameter of the rig ζ_s is taken very close to the same damping parameter as when the main resonance condition occurs in the dynamic system.

```
solNumerical =
NDSolve[Join[system/.{ζs → .00071, a0 → 0.165, Ω0 → 1/2, α → 1.1, γ → 0.1,
B → .2, ε → 1}, {c1(0) == −1.06, c2(0) == −7.91, c3(0) == 0.0, c4(0) == .0}],
{c1(τ), c2(τ), c3(τ), c4(τ)}, {τ, 0, 2900}]//Flatten;
```

As in the previous part of the presentation, examples of the non stationary motion of the specific point in the phase space are presented in the graphics in figure 11 to figure 16.
The wave's motion of the vortex is shown in figure 13 and in figure 14.
The graphics in figure 15 shows the trajectory of the curve $(c_1(\tau), c_2(\tau), c_4(\tau))$ for $\tau \in [100, 2800]$, and the graphics in figure 16 shows the trajectory of the curve $(c_1(\tau), c_2(\tau), c_3(\tau))$ for $\tau \in [100, 2800]$. Both shows periodic behaviour.

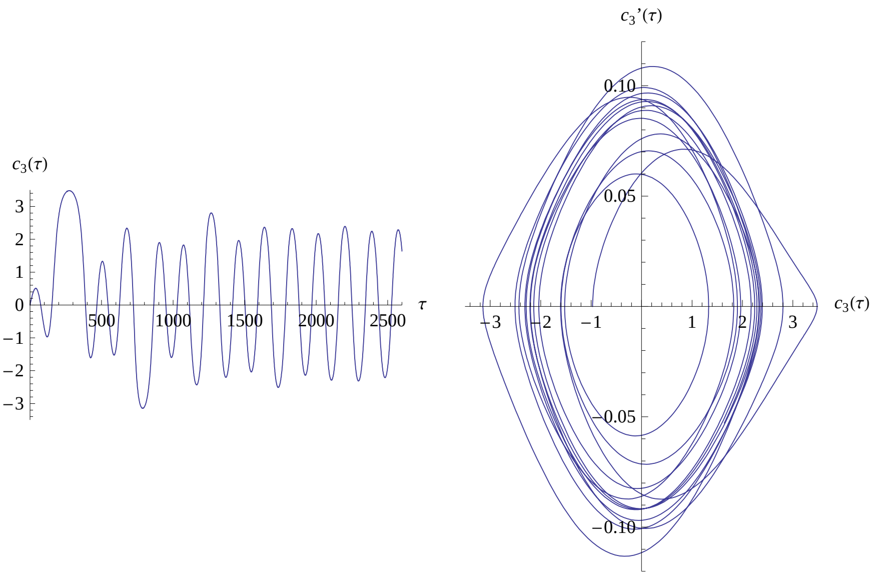

Figure 8: The wave's motion of the vortex. Quasiperiodic motion of $c_3(\tau)$. Here $\zeta_s = 0.001$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_3(\tau)$ as a function of time, on the right, the phasespace of (c_3, c_3').

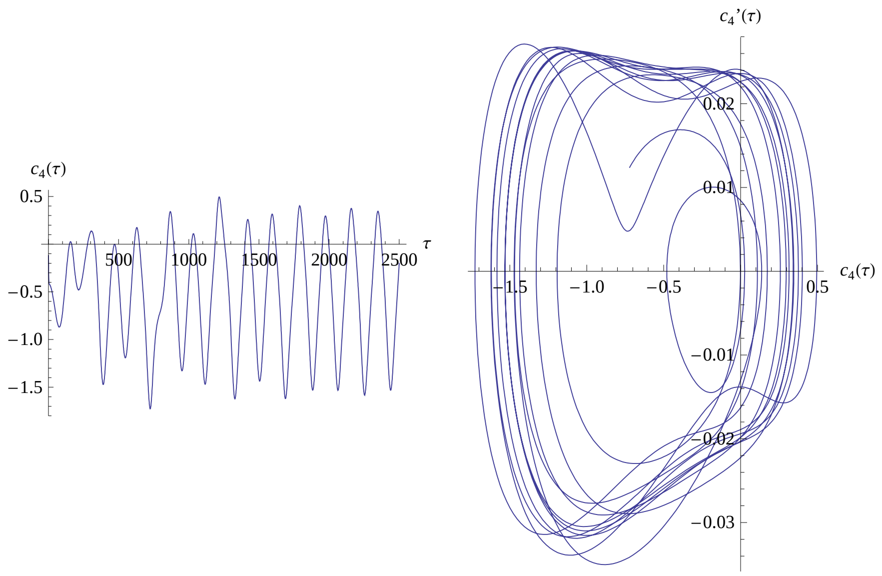

Figure 9: The wave's motion of the vortex. Quasiperiodic motion of $c_4(\tau)$. Here $\zeta_s = 0.001$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_4(\tau)$ as a function of time, on the right, the phasespace of (c_4, c_4').

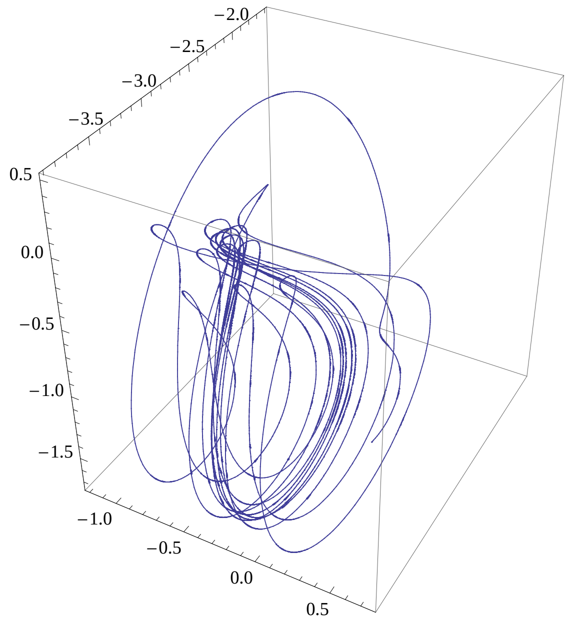

Figure 10: The trajectory of the projection of a curve in 4-space to 3-space, $\pi_{1,2,4}c(\tau) = (c_1(\tau), c_2(\tau), c_4(\tau))$ for $\tau \in [10, 2800]$ for $\zeta_s = 0.0191$ and $\Omega_0 = 1.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. Since each componentfunction is quasiperiodic, the structure of the trajectory of $\tau \to c(\tau)$ is more complicated.

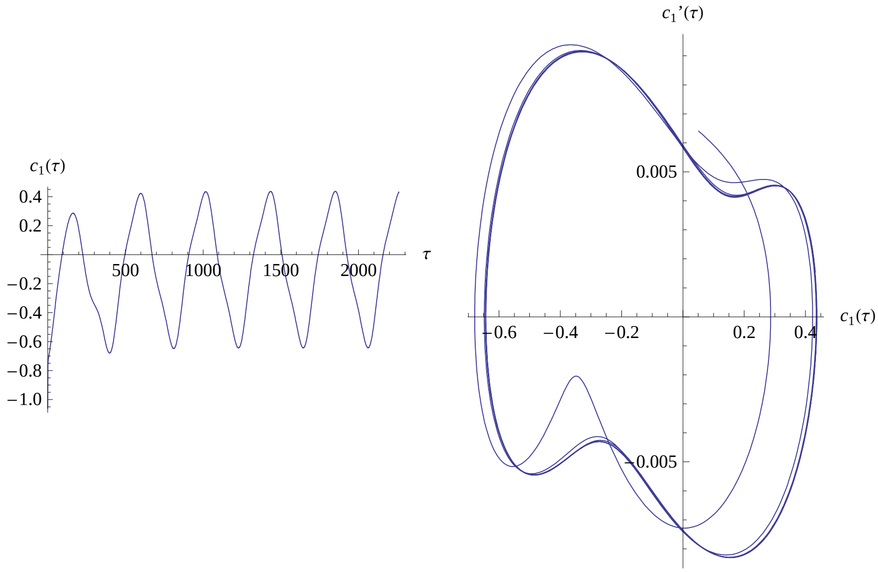

Figure 11: Stable motion of $c_1(\tau)$ to a periodic motion. Here $\zeta_s = 0.00071$ and $\Omega_0 = 0.5$. Here $a_0 = 0.165$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_1(\tau)$ as a function of time, on the right, the phasespace of (c_1, c_1').

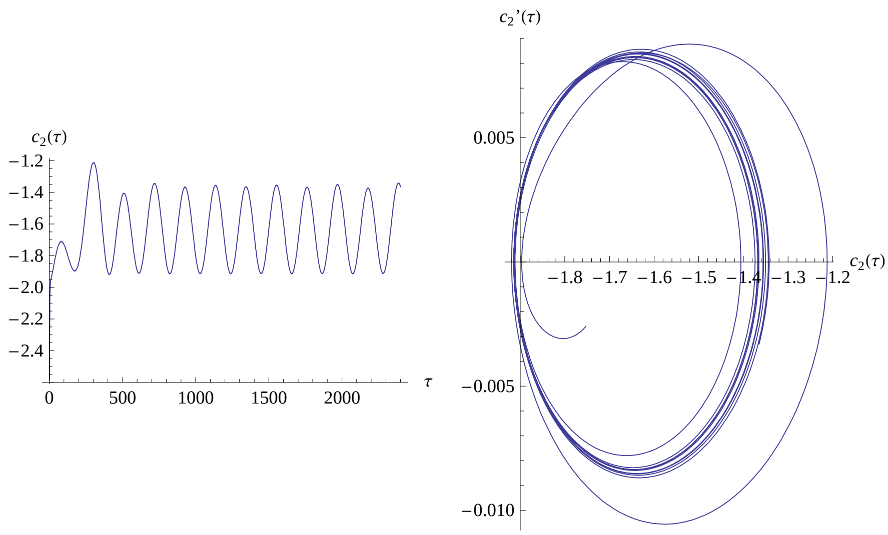

Figure 12: Stable motion of $c_2(\tau)$ to a periodic motion. Here $\zeta_s = 0.00071$ and $\Omega_0 = 0.5$. Here $a_0 = 0.165$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_2(\tau)$ as a function of time, on the right, the phasespace of (c_2, c_2').

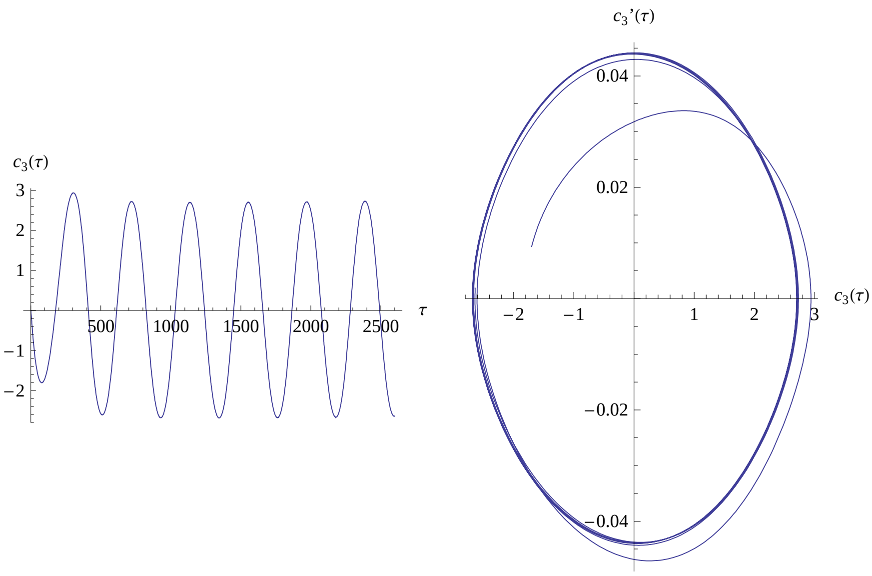

Figure 13: The wave's motion of the vortex. Stable motion of $c_3(\tau)$ to a periodic motion. Here $\zeta_s = 0.00071$ and $\Omega_0 = 0.5$. Here $a_0 = 0.165$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_3(\tau)$ as a function of time, on the right, the phasespace of (c_3, c_3').

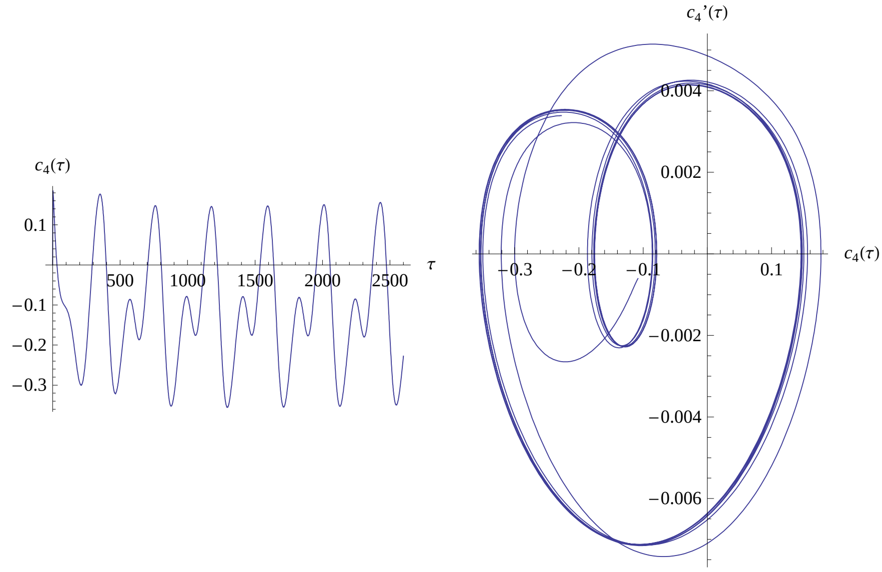

Figure 14: The wave's motion of the vortex. Stable motion of $c_4(\tau)$ to a periodic motion. Here $\zeta_s = 0.00071$ and $\Omega_0 = 0.5$. Here $a_0 = 0.165$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The plot on the left shows the component $c_4(\tau)$ as a function of time, on the right, the phasespace of (c_4, c_4').

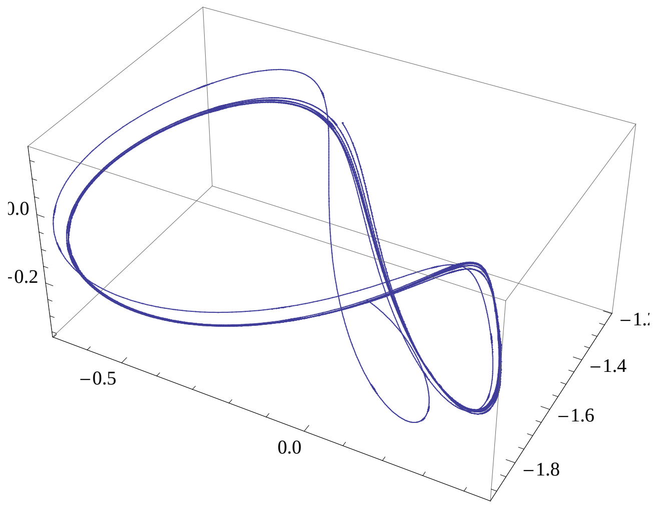

Figure 15: The trajectory of the projection of a curve in 4-space to 3-space, $\pi_{1,2,4}c(\tau) = (c_1(\tau), c_2(\tau), c_4(\tau))$ for $\tau \in [100, 2800]$ for $\zeta_s = 0.00071$ and $\Omega_0 = 0.5$. Here $a_0 = 0.165$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The tracjectory seems to approach a periodic orbit.

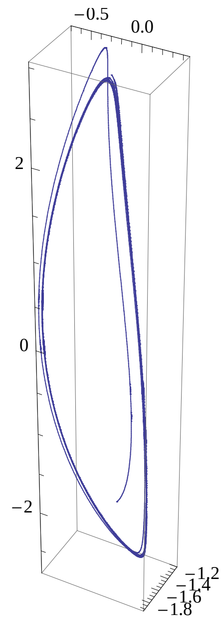

Figure 16: Another projection of the curve $\tau \leftarrow c(\tau)$ as in figure 15. The trajectory of the projection of a curve in 4-space to 3-space, $\pi_{1,2,3}c(\tau) = (c_1(\tau), c_2(\tau), c_3(\tau))$ for $\tau \in [100, 2800]$ for $\zeta_s = 0.00071$ and $\Omega_0 = 0.5$. Here $a_0 = 0.165$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 0.2$ and $\epsilon = 1$. The tracjectory seems to approach a periodic orbit.

Like the results obtained in part 2 the same non-stable amplitudes of the vibrations of the rig are observed at the phase diagrams presented above which deal with a subharmonic resonance in the system.

7. Subharmonic Resonance 1/3. Stability Motion

At least let us consider a numerical solutions of the system of the first approximations when a *subharmonic resonance order 1/3* condition is taking place in the system.

The damping parameter of the rig ζ_s is taken very close to the same damping parameter as in the system when time depending amplitudes are resulting from the simulation of the ODE of the first approximation in the main resonance condition.

solNumerical =
NDSolve[Join[system/.{$\zeta_s \to .000109, a_0 \to .1, \Omega_0 \to 1/3 + 0.001, \alpha \to 1.1,$
$\gamma \to .1, B \to 1.2, \varepsilon \to 1$}, {$c_1$[0]==0.478, c_4[0]==0.0559, c_2[0]==1.313,
c_3[0]==0.22}], {$c_1(\tau), c_2(\tau), c_3(\tau), c_4(\tau)$}, {$\tau$, 0, 2900}]//Flatten;

Unlike to the previous results of simulations a stationary motion of the specific point in the phase space are presented in figures 17 to figure 20 below.
The wave's motion of the vortex is shown in figure 19 and figure 20.
Hence the dynamic behavior of the system depends on the type of resonance is taken place in the system.

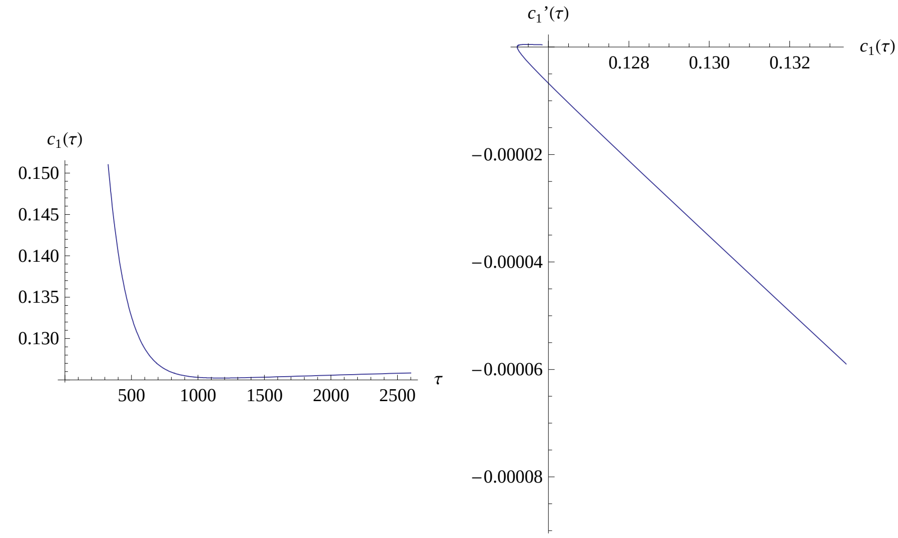

Figure 17: A stationary motion of $c_1(\tau)$ at the 1/3 subharmonic resonance. Here $\zeta_s = 0.000109$ and $\Omega_0 = 1/3 + 0.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 1.2$ and $\epsilon = 1$. The plot on the left shows the component $c_1(\tau)$ as a function of time, on the right, the phasespace of (c_1, c_1').

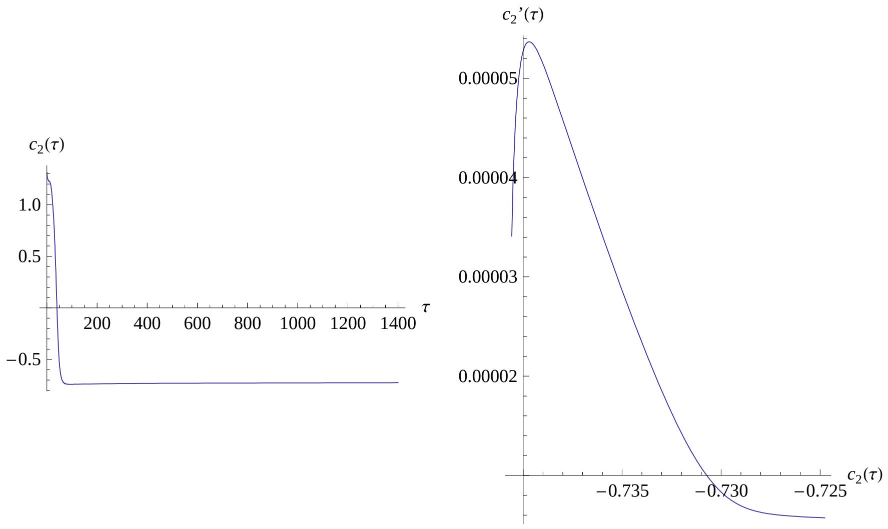

Figure 18: A stationary motion of $c_2(\tau)$ at the 1/3 subharmonic resonance. Here $\zeta_s = 0.000109$ and $\Omega_0 = 1/3 + 0.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 1.2$ and $\epsilon = 1$. The plot on the left shows the component $c_2(\tau)$ as a function of time, on the right, the phasespace of (c_2, c_2').

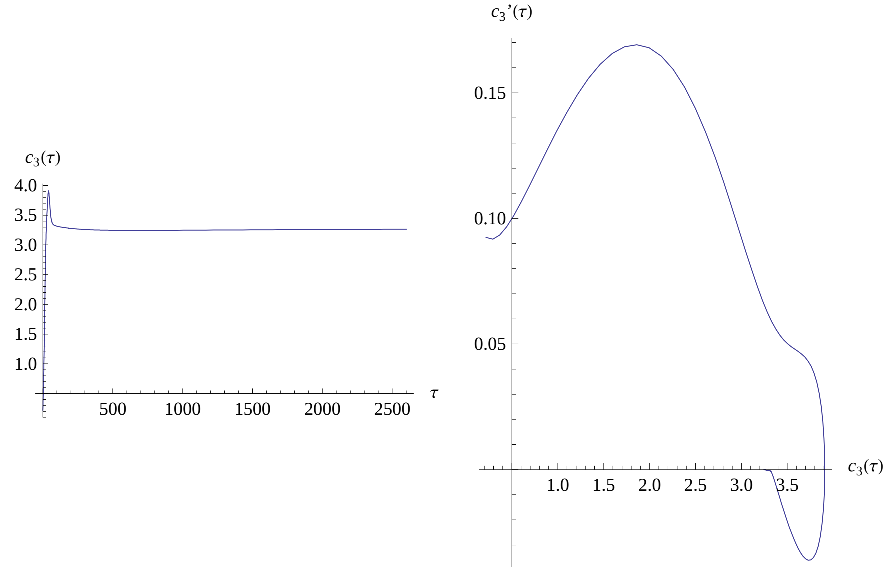

Figure 19: The wave's motion of the vortex. A stationary motion of $c_3(\tau)$ at the 1/3 subharmonic resonance. Here $\zeta_s = 0.000109$ and $\Omega_0 = 1/3 + 0.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 1.2$ and $\epsilon = 1$. The plot on the left shows the component $c_3(\tau)$ as a function of time, on the right, the phasespace of (c_3, c_3').

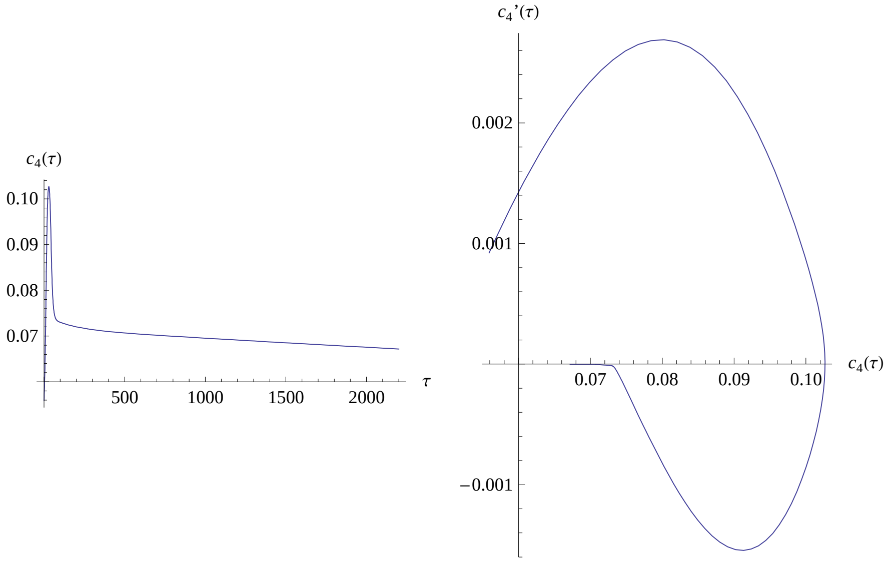

Figure 20: The wave's motion of the vortex. A stationary motion of $c_4(\tau)$ at the 1/3 subharmonic resonance. Here $\zeta_s = 0.000109$ and $\Omega_0 = 1/3 + 0.001$. Here $a_0 = 0.1$, $\alpha = 1.1$, $\gamma = 0.1$, $B = 1.2$ and $\epsilon = 1$. The plot on the left shows the component $c_4(\tau)$ as a function of time, on the right, the phasespace of (c_4, c_4').

Conclusion

The solutions in the time-resolution domain and in the phase-space domain of a two degree of freedom non-linear vibratory system in the notebook are developed by the well-known averaging method using the *Mathematica* symbolic procedure of evaluations.

The phenomenon of resonance instability of the non-linear vibrations of the model of the offshore structure proposed by Sarpkaya and Issacsson [SI81] is obtained and studied in the present paper.

Resonance instability (here it means unregulated time-dependence of the amplitudes of vibrations of the offshore structure) of the offshore structure mainly depends on the type of resonance in the system (main or subharmonic) and on the property of the structural damping of the rig.

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