Modelling of steam turbines for mixed integer nonlinear programming (MINLP) in design and off-design conditions of CHP plants

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Abstract

In this paper mathematical models of steam turbines used in mixed integer nonlinear programming (MINLP) are improved to enable more realistic results from the optimisation and more robust and reliable optimisation models. The modern advanced power plant is in many respects very well optimised, which means that the focus today is on marginal improvements to the process. The models used in the optimisation must be able to reflect this. The challenges of the steam turbine modelling discussed in this paper are the more detailed efficiency modelling of the turbine during the off-design operation, the modelling of pressure as a free variable both in design and off-design optimisation and the possibilities to increase the robustness of the model by convexifying it.

There are several ways of regulating turbines during off-design operation. The condensing turbines in conventional power plants, where the load does not vary or varies very little, are usually regulated with sliding pressure and regulation valves, whereas the back-pressure turbines, *e.g.* in combined heat and power plants, are often regulated using a regulation stage. For power plants where the pressure of superheated steam before the turbine is fixed, the regulation stage improves the steam turbine performance during off-design operation. However, in the optimisation models that exist today the special behaviour of the regulation stage efficiency has not been taken into account. Similarly, the exhaust losses at the end of the turbine affects the efficiency of the turbine at part loads and should be considered in the modelling. A better model of the regulation stage operation and the exhaust losses during the off-design conditions will give a significant contribution to the improvement of the accuracy of the whole MINLP model.

The lack of robustness of the models is a problem in steam turbine modelling as the more detailed turbine modelling easily results to an increased amount of nonlinearities and nonconvexities in the optimisation model. With the nonconvex model the solution of the optimisation model may be problematic with commercial solvers, optimality of the solutions can be guaranteed only locally and the solutions can be sensitive to the initial values.

In this work the accuracy of steam turbine models for MINLP optimisation are improved by including the special behaviour of the regulation stage efficiency and the exhaust losses at the end of the turbine to the model. In addition, the optimisation of the pressure levels in the turbine during design and off-design operation is possible as the pressures are modelled as free variables. At the same time the nonconvexities of the models are reduced by estimating the nonconvex functions with convex functions. The results show that the developed nonconvex model with the detailed efficiency functions and the free pressure variables is a more accurate description of the simulated steam turbine than the models used previously. Also, the convex model based on this gives a good description of the behaviour of the total steam turbine. The developed models will be especially useful, when a larger MINLP model of a whole CHP plant is constructed.

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Nomenclature

A	Area	(m^2)
a	Coefficient	
b	Coefficient	
c	Coefficient, Velocity vector	(-), (m/s)
h	Specific enthalpy	(kJ/kg)
k	Flow-through factor	
m	Mass flow	(kg/s)
n	Polytropic exponent	
p	Pressure	(bar)
Q	Heat flow	(MW)
R^2	Coefficient of determination	
s	Specific entropy	(kJ/kgK)
T	Temperature	$(K, {}^\circ C)$
U	Overall heat transfer coefficient	(W/m^2K)
v	specific volume	(m^3/kg)
W	Work	(MW)
x	Steam quality	
Z	Convex transformation	
η_{is}	Isentropic efficiency	

1 Introduction

Mathematical programming, or optimisation, deals with the problem of optimising an objective function. A general mixed integer nonlinear programming (MINLP) model can be formulated as:

$$\min Z = f(x,y) \quad \text{s.t.} \begin{cases} h(x,y) = 0 \\ g(x,y) \leq 0 \\ x \in X, y \in \{0,1\}^m \end{cases} \tag{1}$$

where $f(x,y)$ is the objective function, $h(x,y) = 0$ are equations that can describe for instance energy- and mass balances and $g(x,y) \leq 0$ are inequalities, that for example can describe the feasible operation points (*e.g.* temperature ranges). A good overview of optimisation and applications in process systems engineering can be found in the work by Biegler and Grossmann [2] and Grossmann and Biegler[9].

There are several important factors to take into account when formulating an optimisation model, for instance the convexity of the functions, existence of derivates, continuity of the functions and the linearity or the nonlinearity of the functions. There is no efficient algorithm for solving problems of all classes, but many specialised algorithms have been developed. For linear programming (LP) problems, and to some extent mixed integer linear programming (MILP) problems, there exists powerful algorithms to solve practical problems to a deterministic global optimum. For the nonlinear problems, there exist algorithms to solve both nonlinear programming (NLP) and MINLP problems to global optimal solutions. A good overview of global optimisation can be found in the work by Floudas et al. [8]. However, the most efficient algorithms require that the functions are convex in order to guarantee a global optimal solution. Broadly speaking, a linear model is better than a nonlinear, continuous variables are better than binary/integer variables and convex functions are better than concave/nonconvex functions. For nonconvex problems with multimodal objective function or nonconvex feasible region, the classical nonlinear programming algorithms will terminate with a solution, which is strongly dependent on the starting point. Unfortunately, these problems are common in the design and synthesis of energy systems. This means that for these models special care must be taken to make them suitable for the mathematical programming solvers.

Combined heat and power (CHP) plants are power plants that generate heat as well as electricity. Today they are an important part of the modern energy conversion chain. A CHP plant can generate heat for both district heating and industrial processes. Most CHP plants are operating under time varying constraints. Process conditions like heat and power demand, the electricity spot-price and fuel composition are very important time varying factors for optimal design and operation of CHP plants with respect to both profit and environmental impact of the process. In recent years carbon dioxide management together with emission trading has also become more and more important. The largest responsibility for complying with the emission limits set by the Kyoto Agreement are given to the energy industry. The modern advanced power plant is in many respects very well optimised, which means that even marginal improvements to the process are important. The models that are developed to design plants or to suggest and evaluate improvements must be able to reflect this. In this respect the models of the steam turbines included in the CHP plant models are important.

In addition to the fact that the steam turbines are crucial to the operation of the CHP plant, the steam turbine characteristics have several features that can make the models difficult to solve for the current mathematical programming solvers. The difficulties are primarily related to the nonconvexities of the models. A discussion of some of these aspects can be found in the work by Tveit [21].

In the literature there are many different models of steam turbines. For relatively simple steam turbines, *i.e.* turbines with few extractions, that are regulated using regulation valves, there exists a good linear model that relates the power output of the steam turbine to the mass flow of steam through the turbine. This relation is often referred to as the *Willan's line* [6] [11]. The mathematical form of the Willan's line is shown in Equation 2.

$$W_{out} = \sum_{i=0}^{1} c_i \cdot m^i \tag{2}$$

where W_{out} is the power generated by the steam turbine, c_i are coefficients and m is the mass flow of steam through the turbine.The Willan's line is approximately a straight line between the smallest load, when the turbine is running but not yet producing power, and the load with the maximum efficiency. The Willan's line has been used in optimisation for instance by Movromatis and Kokossis [14] [15] and by Manninen and Zhu [12]. However, the Willan's line does not give a good model of more complex turbines and fails to take into account the nonlinearities related to the regulation stage and exhaust loss (see the work by Savola and Keppo [19] for a discussion). The modelling of regulation stages and exhaust losses are discussed in Section 2.

An optimisation model formulation for a CHP plant has previously been presented by Bruno et al. [3]. In their formulation the complexity of the model was reduced by fixing the steam pressures in the model. The steam turbine efficiency was calculated using linear and quadratic correlations depending on the extraction pressure.

In this paper steam turbine models with Willan's line, with more detailed regulation stage and exhaust loss efficiency curves and with convexified functions are presented and compared with a simulation model. The basic model formulation is different to the formulations by Bruno et al., as model is multiperiod and the pressures in the models are free variables. This requires more complex models but makes it possible to define *e.g.* the optimal extraction pressures and the part load operation, which are especially important when the steam turbine models are combined to larger CHP plant models.

2 Steam turbine characteristics

For mathematical programming problems related to the optimisation of CHP plants, there are certain characteristics of the steam turbine that are important to model. Basically, a good model must be able to calculate the mass- and energy balances for the steam turbines. Expressions for the mass- and energy balances for a steam turbine stage are given in Equations 3 and 4 respectively.

$$\sum_{i\in IN} m_i - \sum_{j\in OUT} m_j = 0 \tag{3}$$

$$\sum_{i\in IN} h_i \cdot m_i - \sum_{j\in OUT} h_j \cdot m_j - W = 0 \tag{4}$$

where m is the steam mass flow, h is the steam enthalpy which is a function of temperature and pressure. W is the power output of the stage and can be written as $W = \eta_{is} \cdot m \cdot (h_{in} - h_{out,is})$. In terms of enthalpies, the isentropic efficiency can be written as shown in Equation 5.

$$\eta_{is} = \frac{h_{in} - h_{out}}{h_{in} - h_{out,is}} \tag{5}$$

where h is the enthalpy and h_{is} is the enthalpy of an isentropic process. The way the isentropic efficiency varies with the mass flow is determined by the design of the turbine stage. The isentropic efficiency varies differently if the stage is a regulation stage, a working stage or the last stage of the turbine. The isentropic efficiency, η_{is}, is a function of the load or mass flow through the turbine stage. This variation of the efficiency of steam turbines is especially important for CHP plants compared to conventional condensing power plants, as CHP plants normally operate at partial loads for a considerable time during a year.

A regulation stage can be used to improve the off-design efficiency of a steam turbine compared to regulation using a regulation valve. A regulation stage is a group of nozzles that separates sectors of an impulse turbine stage. The nozzle group is regulated with special valves that controls the flow of steam to each sector. The difference in the steam expansion in an enthalpy-entropy diagram for a steam turbine with a regulation valve compared to a regulation stage is shown in Figure 1. The efficiency of the regulation stage is typically designed to be at its maximum at partial steam load (90% load). In this work the part load performance of the regulation stage is defined as a polynomial function, which is based on the estimation that the maximum efficiency of the regulation stage, 0.80, is gained at around 90% steam load, corresponding to the characteristics of a typical turbine. At full load the efficiency is 0.75 and as the steam load decreases towards 10% the efficiency goes to zero. This estimation, where the maximum efficiency of the steam turbine is gained at part load, corresponds the usual conditions in a CHP plant. With these estimations the regulation stage efficiency starts to decrease more rapidly, when the steam load is less than 80-70 %. Figure 2 shows how the efficiency of a typical regulation stage changes with the load.

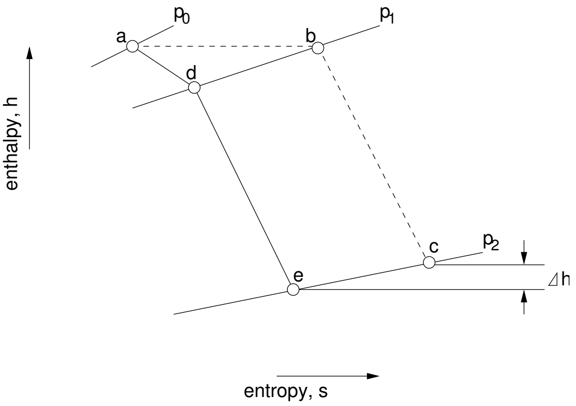

Figure 1: Difference between the steam expansion in a regulation valve (*abc*) and regulation stage (*ade*). The additional power obtained when using a regulation stage is Δh, which is the difference in enthalpy between the states c and e.

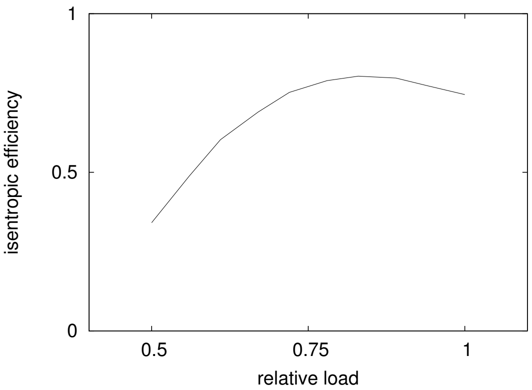

Figure 2: The isentropic efficiency, η_{is}, as a function of relative load (mass flow of steam through the turbine) for a regulation stage.

The so-called turbine working stages after the regulation stage are most often reaction stages with a degree of reaction of 0.5. The isentropic efficiency as a function of the load for a typical working stage is shown in Figure 3. As can be seen from the figure, the change in the isentropic efficiency is small and close to linear between 50% and 100% of the load at the design stage. Below 50% of the load, the isentropic efficiency starts to behave nonlinearly.
The isentropic efficiency of the last turbine stage will have a different behaviour than the previous stages.

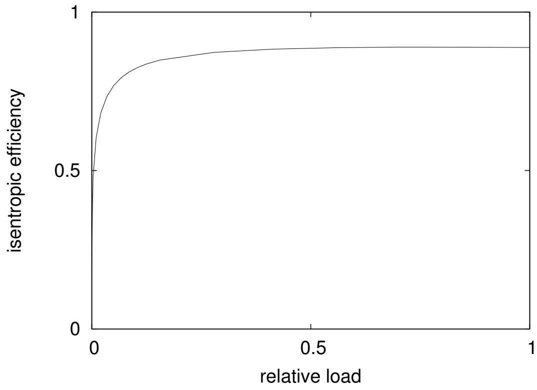

Figure 3: Isentropic efficiency, η_{is}, as a function of the load relative to the design state for a typical steam turbine working stage. The isentropic efficiency changes approximately linearly with the load from the design state to about 50% of the load, after which the isentropic efficiency starts to deteriorate rapidly.

Steam exiting from the last stage of the turbine will inevitably have kinetic energy that is not utilised in the turbine and thus lost, this is called the *exhaust loss*. The exhaust loss affects the off-design variations of the isentropic efficiency of the turbine stage. It is thus important to be able to calculate the exhaust loss, since it has a major impact on the power output of the turbine stage. In Figure 4 the typical exhaust loss behaviour for a turbine stage is shown. The calculation of the last stage is also complicated by the fact that the steam starts to condense in the turbine stage. This means that the steam quality must be taken into account by the model. The steam quality, x, can be defined for the isentropic case as shown in Equation 6.

$$x = \frac{s_{in} - s_{out}^{water}}{s_{out}^{steam} - s_{out}^{water}} \tag{6}$$

where s is the entropy.
In order to be able to calculate the enthalpy change of the steam and the temperature of the steam being extracted, it is necessary to calculate the pressure at the inlet and outlet of a turbine stage. Equation 7 shows an expression of the relationship between the mass flow of steam and the inlet and outlet pressures of a steam turbine stage with a fixed blade construction [20].

$$\left(\frac{m}{m_0}\right)^2 = \frac{\overline{\mu}^2 p_\alpha v_{\alpha 0}}{\overline{\mu}_0^2 p_{\alpha 0} v_\alpha} \left(\frac{1 - \frac{p_\omega}{p_\alpha}^{\left(\frac{n+1}{n}\right)}}{1 - \frac{p_{\omega 0}}{p_{\alpha 0}}^{\left(\frac{n+1}{n}\right)}}\right) \tag{7}$$

where the subscript 0 refers to the design state, n is the polytropic exponent and $\mu = k_2 c_{n2}/\sqrt{2\Delta h_s}$. c_{n2}

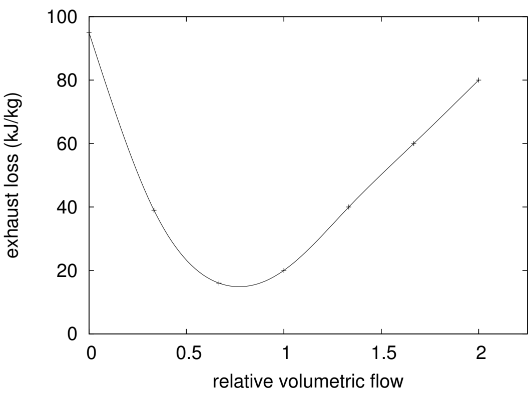

Figure 4: Typical exhaust loss behaviour for a turbine stage as a function of the relative volumetric flow of steam (based on data from Prosim module reference manual [7]). The volumetric flow is relative to the volumetric flow at the design state. The exhaust loss is dependent on among others the blade length, mean diameter, frequency and outlet angle of the turbine stage [17].

is the normal component of the velocity vector of the steam at the exit, Δh_s is the isentropic enthalpy difference for the stage and k_2 is the flow-through factor. If the CHP plant is generating steam to an industrial process, there also is another reason why it is important to calculate the pressure. Industrial processes have often lower bounds for the pressure of the steam, and the only way this can be accurately modelled is to calculate the changes in pressures.

To summarise, in order to calculate realistic energy- and mass balances, it is necessary to know the mass flow of steam through the turbines, the pressures and temperatures before and after the turbine stages and the isentropic efficiencies of the steam turbines. For CHP plants the heat demand determines the operation of the plant. As the heat demands vary so do the mass flow of steam through the turbine, the pressures in the turbine and the isentropic efficiency. A good model must thus relate the heat demand to these values.

In order to get a model of the steam turbine it is necessary to relate the heat demand of the CHP plant to the mass flow of steam. The heat demand of the CHP plant can be related to the mass flow of steam with the simple model of a heat exchanger shown in Equations 8 and 9.

$$Q = m \cdot \big(h(T_L, p_L) - h(T_0, p_0) \big) \tag{8}$$

$$Q = U \cdot A \cdot \Delta T_{lm} \tag{9}$$

where m is the mass flow of steam, h is the enthalpy, Q is the heat load of the heat exchanger, U is the overall heat transfer coefficient which is assumed to be constant along the heat exchanger and A is the heat exchanger surface area. The indices 0 and L refers to the start and end of the heat exchanger, where in this case the hot steam enters the heat exchanger at L and exits at 0. ΔT_{lm} is the *logarithmic mean temperature difference* (LMTD).

The LMTD has certain unwanted properties with regards to mathematical programming, and it is therefor common to use an approximation of LMTD that is more suitable, *e.g.* the approximations suggested by Chen [4] and Paterson [18]. In this work the Chen-approximation is used, and the expression can be found in Appendix A.

Another important aspect to model is the physical properties of water and steam. For the work presented in this paper the most important properties are the enthalpy and entropy of water and steam as a function of temperature and pressure. A discussion of steam property functions and mathematical programming can be found in the work by Laukkanen and Tveit [10]. The property functions used in this work are adapted from the Industrial Standard IAPWS-IF97 [22].

Many of the equations presented above have properties that cause nonconvexities to the optimisation model and thus make it more difficult to find the global optimum of the problem. Section 3 summarises, how these nonconvexities are transferred into convex formulations in the steam turbine model.

3 Convex NLP model formulation of a steam turbine

The bilinear term in the energy balance in Equation 4, namely the mass flow times the enthalpy, can be approximated by the convex envelope suggested by McCormick [16]. The expression for the convex envelope is shown in Equation 10.

$$m \cdot h \approx \begin{cases} m^L h + h^L m - m^L h^L & \text{if h} \leq -z \\ m^U h + h^U m - m^U h^U & \text{otherwise} \end{cases} \tag{10}$$

where m^L, m^U, h^L and h^U are the upper and lower bounds of the mass flow and enthalpy respectively. The breakpoint, z, is defined in Equation 11.

$$z = \frac{h^U - h^L}{m^U - m^L} m + \frac{m^U h^U - m^L h^L}{m^U - m^L} \tag{11}$$

In order to avoid discontinuities in the optimisation model, the bilinear term can be replaced by a variable

ω, which relaxes the upper and lower bound.

$$\omega \geq m^L h + h^L m - m^L h^L \quad (12)$$

$$\omega \geq m^U h + h^U m - m^U h^U \quad (13)$$

$$\omega \leq m^U h + h^L m - m^U h^L \quad (14)$$

$$\omega \leq m^L h + h^U m - m^L h^U \quad (15)$$

where Equations 12 and 13 comprise the relaxed lower bound and respectively Equations 14 and 15 the upper bound.

The maximum difference between the bilinear term $m \cdot h$ and the convex envelope in Equation 10 was shown by Androulakis et al. [1] to be $\frac{(h^U-h^L)(m^U-m^L)}{4}$ and occurs at the middle point, *i.e.* when $m = \frac{m^L+m^U}{2}$ and $h = \frac{h^L+h^U}{2}$.

Similarly to the energy balance, the bilinear terms with the steam content x after the turbine, $x \cdot h$, $x \cdot s$, $x \cdot h_{is}$ and $x \cdot m$, are approximated using the convex envelope formulation.

If the load does not go below 50% of the design load, the isentropic efficiency for a regular working stage can be accurately modelled as constant. For the regulation stage and the last stage, the variations of the isentropic efficiency with the load is significant. The isentropic efficiency for the regulation stage as a function of load can be accurately modelled using a 3rd degree polynomial. As can be easily understood by looking at Figure 2, the 3rd degree polynomial is concave in the relevant region between the loads 100% and 50%. The concave polynomial can, however, easily be convexified at the cost of an additional equation and variable. For each period, j, the isentropic efficiency, η_{is}^{reg}, for the regulation stage can be calculated using the convex model in Equations 16 and 17.

$$\eta_{is,j}^{reg} = -Z_{\eta_{is}^{reg},j} \quad (16)$$

$$Z_{\eta_{is}^{reg},j} = \sum_{i=0}^{3} a_i \cdot (Q_j^{rel})^i \quad (17)$$

where $Z_{\eta_{is}^{reg},j}$ is the negative of the isentropic efficiency of the regulation stage, a_i are the coefficients for the polynomial and Q_j^{rel} is the relative heat load for the period j. Similarly, the isentropic efficiency for the last stage, η_{is}^{last} can be modelled by a 6th degree concave polynomial. The convexified expression is shown in Equations 18 and 19.

$$\eta_{is,j}^{last} = -Z_{\eta_{is}^{last},j} \quad (18)$$

$$Z_{\eta_{is}^{last},j} = \sum_{i=0}^{6} b_i \cdot (Q_j^{rel})^i \quad (19)$$

where b_i are the coefficients for the polynomial. The isentropic efficiency and enthalpy Equation 5 for regulation stage and the last turbine stage is approximated by using the convex envelope formulation of the bilinear terms $Z_{\eta_{is}^{last},j} \cdot h$, $Z_{\eta_{is}^{last},j} \cdot h_{is}$, $Z_{\eta_{is}^{reg},j} \cdot h$ and $Z_{\eta_{is}^{reg},j} \cdot h_{is}$. Under the assumptions that the polytropic exponent, n, is constant and equal to 1 and μ is constant and that the steam is ideal steam (i.e. $p_\alpha v_\alpha = p_{\alpha 0} v_{\alpha 0}$), Equation 7 can be written as:

$$p_\alpha^2 m_0^2 - m^2 p_{\alpha 0}^2 = p_\omega^2 m_0^2 - m^2 p_{\omega 0}^2 \quad (20)$$

Equation 20 can be transformed into a bilinear form by replacing the quadratic terms with new variables of the form $Z_{x^2} = x^2$.

$$Z_{p_\alpha^2} Z_{m_0^2} - Z_{m^2} Z_{p_{\alpha 0}^2} = Z_{p_\omega^2} Z_{m_0^2} - Z_{m^2} Z_{p_{\omega 0}^2} \quad (21)$$

This bilinear equation can be approximated by a convex envelope similar to Equation 11.

In the models the approximation of $LMTD$ by Chen [4] is used. The approximation is concave (see Appendix A for proof), which means that it can be transformed into a convex expression using the same transformation as for Equations 16 and 18, namely that $LMTD_{Chen} = -Z_{LMTD_{Chen}}$ and $Z_{LMTD_{Chen}} = -$(concave expression). Equation 9 is approximated by replacing the bilinear term $A \cdot LMTD$ with a convex envelope.

4 Comparison between nonconvex, convex, Willan's line and simulation models

The steam turbine selected for the simulation and modelling case is a back-pressure turbine producing 16.5 MW district heating and 6.1 MW electricity. The turbine consists of three turbine modules as presented in Figure 5. The first turbine module is a regulation stage, the second is a working turbine and the last stage includes the exhaust losses at the end of the turbine. A similar decomposition of a steam turbine into the modules corresponding the expansion between the steam extractions has been presented by Chou and Shih [5]. The superheated steam temperature before the regulation stage of the turbine is 510°C, the pressure 60 bar and the mass flow of steam 7.8 kg/s. The temperature and pressure before the turbine are constant also at part loads. The steam extraction from the working turbine is 5% of the total mass flow. The district heat demand varies from 100% to 50% during the part load operation and defines the steam mass flows, the temperatures and the pressures in the turbine. In the previous

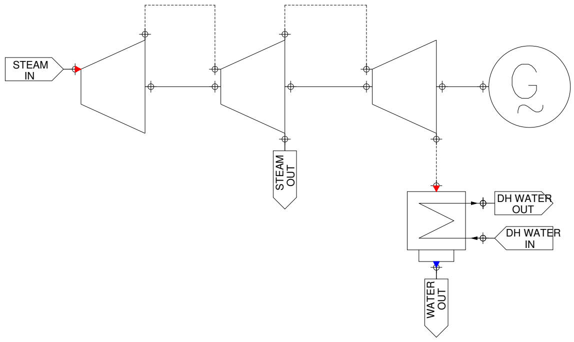

Figure 5: Demonstration turbine.

studies the possible minimum load for a general biofuelled steam turbine system is mentioned to be as low as 30% (Marbe et al. [13]). At lower loads, the boiler will have difficulties supplying the required steam. Usually, the small-scale ($< 20\ MW_e$) CHP plants are operated from full loads down to 50% load, so this is set as the lower limit for the load.
The selected steam turbine is modelled in four different ways.

1. A simulation model is constructed with a simulation program Prosim including the part load behaviour presented in Figures 2 and 4 for the regulation stage and the last turbine stage. The efficiency of the working stage is estimated to remain constant at the loads between 100% to 50%.

2. A NLP optimisation model of the steam turbine is constructed using the Willan's line as a description of the total steam turbine efficiency.

3. The NLP model is modified so that the efficiencies of the regulation stage, the working stage and the last stage are modelled as described in Figures 2, 3 and 4.

4. The nonconvex NLP model is convexified by using *e.g.* the convex envelope method by McCormick [16] and transformations. However, in the convexified steam turbine model the steam property functions remain nonconvex as their convexification is beyond the scope of this work.

The coefficients of the Willan's line, c_i, and the convex NLP model, a_i and b_i, for Equations 17 and 19 respectively are given in Appendix B. It should be noted that the coefficients of the nonconvex model are $-a_i$ and $-b_i$ as shown in Equations 16 and 18.
The optimisation program used is the General Algebraic Modeling System (GAMS) by GAMS Development Corporation and the solver selected for the problem is CONOPT3 by ARKI Consulting and Development A/S. The number of equations and variables and the solving times for the three optimisation models are presented in Table 1. From the table it can be clearly seen that the convexifying of the steam turbine model is done at the costs of more variables and equations in the model and thus also a longer solution time.

	Variables	Equations	Solving time (s)
Willan's line	587	519	11.5
Nonconvex	609	539	16.5
Convex	1215	2327	41.4

Table 1: Statistics regarding the size and complexity of the different optimisation models.

The relative power generation of the total steam turbine system as a function of the relative heat loads according to the optimisation models are presented in Figure 6.

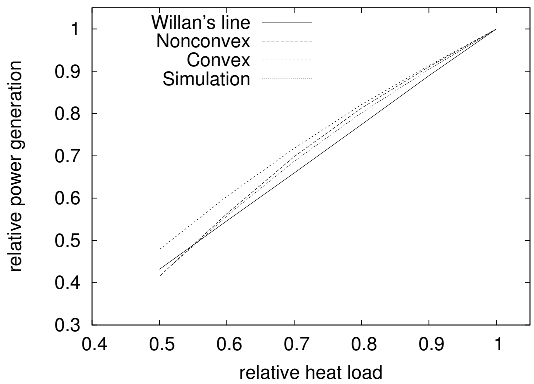

Figure 6: Relative power production of the steam turbine as a function of the relative heat demand according to the tested models.

The results show that when the nonlinear behaviour of the steam turbine efficiency is included into the non-convex optimisation model, it describes the behaviour of the simulated case plant more accurately than the linear description estimated with the Willan's line. The non-linear convex optimisation model deviates more from the simulated values than the Willan's line based model, especially at low loads. However, the convex model gives a better description of the trend than the Willan's line based model. The nonconvex NLP model gives slightly higher relative power productions than the simulation model. The reason for this may be that in the optimisation models the power production as an objective function was maximised and thus the models may find more optimal process designs than the simulated one. The convex model gives higher power productions for the total steam turbine system than the nonconvex one. This is due to the convex envelope method that is used to replace the accurate bilinear nonconvex functions with upper and lower limits of the bilinear term. In the convex envelope method the variables are allowed to vary inside this predefined envelope. To make the convex model more similar to the nonconvex model the envelopes could have been divided to several piecewise envelopes. This would require the use of binary variables. However, in this case the convex model was an accurate enough description of the nonconvex one especially when considering the power generation of total steam turbine system.

The differences of the models can be studied more detailed if the relative power generation as a function of the relative heat load is presented separately for each turbine stage as in Figures 7, 8 and 9. As the Willan's line is a description of the efficiency of the whole steam turbine system, the model with Willan's line is not included into these figures. The results in Figures 7, 8 and 9 show that the nonconvex NLP model is a good description of the simulated process. The differences between the convex model and the nonconvex one became more apparent for the separate turbine stages than for the total steam turbine model. The major difference between the convex and the nonconvex models is in the performance of the last turbine stage. This is probably due to the fact that there are more bilinear terms, related to the steam moisture content after the turbine and to the heat exchanger design, that are replaced with convex envelopes affecting the last stages then the rest of the turbine. Thus to improve the convex model the formulation of the last stage behaviour should be transferred into more accurate one.

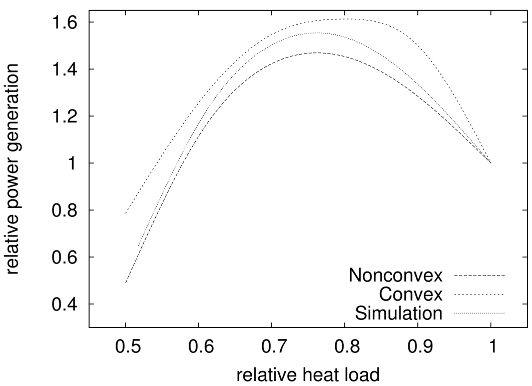

Figure 7: Relative power production of the regulation stage as a function of the relative heat demand according to the tested models.

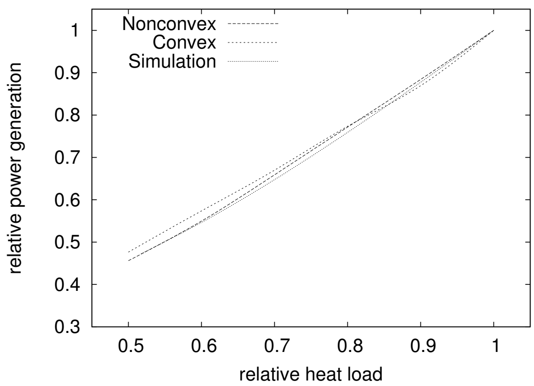

Figure 8: Relative power production of the 2nd turbine stage (working stage) as a function of the relative heat demand according to the tested models.

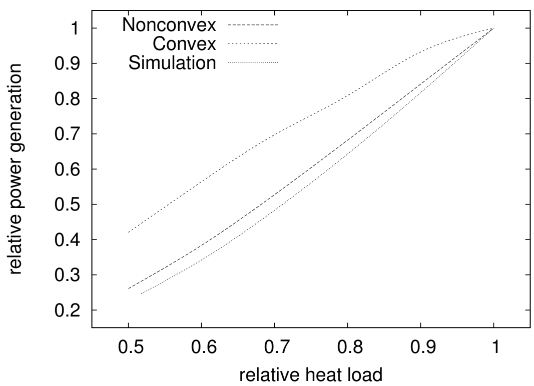

Figure 9: Relative power production of the last turbine stage as a function of the relative heat demand according to the tested models.

The developed models have also been compared by calculating the coefficients of determination, R^2, between the results of the optimisation models and the simulation models. The R^2 values describe how well the variations in the results from the optimisation models explains the variations in the results from the simulation model. The R^2 values are presented in Table 2. According to the R^2 values the nonconvex model is the best presentation of the simulated steam turbine. Also the behaviour of the different turbine stages in the nonconvex model correspond well with the simulation model. The Willan's line and the convex model are also fairly good descriptions of the total steam turbine behaviour, but the Willan's line doesn't model the turbine stages separately and the convex model gives

	Regulation stage (Fig. 7)	Working turbine (Fig. 8)	Last stage (Fig. 9)	Total turbine (Fig. 6)
Willan's line	-	-	-	0.9896
Nonconvex	0.9111	0.9966	0.9864	0.9955
Convex	0.9044	0.9942	0.5991	0.9812

Table 2: The coefficients of determination, R^2, for the results from the different optimisation models compared to the results from the simulation model.

a poor correlation to the last turbine stage. If a more accurate convex model is needed it would be possible to improve the modelling of the last stage *e.g.* by dividing the used convex envelopes into the smaller subregions. However, for a MINLP modelling needs of a CHP plant the convex model gives a good enough description already in its current state especially if the total steam turbine behaviour is the main interest of the optimisation.

5 Conclusions

In this paper a nonconvex and a convex NLP optimisation model was developed for a steam turbine operating in a CHP plant. The purpose of the work was to provide more accurate models describing the special behaviour of the steam turbine that could be included into the larger MINLP models of the CHP plants both in design and off-design optimisation. In previous steam turbine models included into the CHP models the behaviour of the steam turbine is often described with a Willan's line and the pressure variables in the models have been fixed to reduce the complexity of the problem. In the nonconvex and convex models presented here the Willan's line is substituted with more accurate efficiency curves describing the regulation stage efficiency and the exhaust losses at the end of the turbine. Also, a method to keep pressures as free variables in the model during the simultaneous design and off-design optimisation is presented in the paper and included into the model.

The results of the nonconvex and convex optimisation models are compared with the results from the similar optimisation model based on the Willan's line and with a simulation model of an existing steam turbine. The results show that the nonconvex model is the best description of the total simulated steam turbine as well as of the behaviour of the separate steam turbine stages. The convex model and the Willan's line are also describing the behaviour of the total steam turbine system well, but the detailed turbine stage modelling is less accurate. The Willan's line is a description of only the total turbine efficiency and does not take into account the different turbine stages. The convex model describes fairly well the regulation stage and the working turbines behaviour but the model for the last stage is less accurate. This is due to the fact that the last stage is influenced by a large amount of convex envelope functions, which are used to replace the nonconvex bilinear functions related to the moisture content of the steam after the turbine. However, the convex description of the total steam turbine behaviour may be very useful when a larger MINLP model of a CHP plant is constructed. A convex model of the steam turbine may be preferred as it helps to formulate the larger model in a convex way, which makes the model more robust and the finding of the global optimum easier.

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A Proof that the Chen approximation is concave

The Chen approximation [4] can be written as:

$$LMTD_{Chen} = \left(\Delta T_1 \Delta T_2 (\frac{\Delta T_1 + \Delta T_2}{2})\right)^{\frac{1}{3}} \tag{22}$$

where

$$\Delta T_1 = T_{in}^{hot} - T_{out}^{cold} \tag{23}$$

$$\Delta T_2 = T_{out}^{hot} - T_{in}^{cold} \tag{24}$$

Since $T_{in}^{hot} \geq T_{out}^{cold}$ and $T_{out}^{hot} \geq T_{in}^{cold}$, then $\Delta T_1, \Delta T_2 \geq 0$. The principle minors of the Hessian of Equation 22 are presented in Equation 25 and 26.

$$\text{principle minor}_1 = 0 \tag{25}$$

$$\text{principle minor}_2 = -\frac{f \cdot g}{h} \tag{26}$$

where

$$f = (\Delta T_1^2 \Delta T_2)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}} \tag{27}$$

$$g = 2\Delta T_1^2 \Delta T_2^2 + \Delta T_2^4 + \Delta T_1 \Delta T_2^3 + \Delta T_1^4 + \Delta T_1^3 \Delta T_2 \tag{28}$$

$$h = 9\Delta T_1^2 \Delta T_2^2 (\Delta T_1^2 + 2\Delta T_1 \Delta T_2 + \Delta T_2^2) \tag{29}$$

From Equations 27, 28 and 29 it can easily be seen that $f, g, h \geq 0 \ \forall \Delta T_1, \Delta T_2 \geq 0$, which means that $-\frac{f \cdot g}{h} \leq 0 \ \forall \Delta T_1, \Delta T_2 \geq 0$. This means that both principle minors of the Hessian matrix of Equation 22 are less than or equal to zero. Consequently Equation 22 is negative semidefinite and thus concave.

B Coefficients

i	0	1	2	3	4	5	6
a_i	3.89	-13.90	13.02	-3.85			
b_i	-1.06	3.62	-10.01	8.04	3.12	-7.21	2.64
c_i	-660.60	322.58					